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THE
CALCULUS OF OBSERVATIONS

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THE CALCULUS OF OBSERVATIONS

A Treatise on Numerical Mathematics

BY

SIR EDMUND WHITTAKER
LL.D., Sc.D., F.R.S.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF EDINBURGH

AND

G. ROBINSON
M.A., B.Sc.
FORMERLY LECTURER IN MATHEMATICS IN THE UNIVERSITY OF EDINBURGH

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PREFACE

THE mathematical problems which arise in dealing with numerical data are attractive and important. Some knowledge regarding them is required by workers in many different fields—astronomers, meteorologists, physicists, engineers, naval architects, actuaries, biometricians, and statisticians, as well as pure mathematicians; but until recently there has been very little instruction on the subject in the mathematical departments of the British universities. Of late, however, there has been a great awakening of interest in it; and it is now included in the syllabus for the Open Competitive Examination for appointments in the Home and Indian Civil Services, the Colonial Service, etc.

The present volume represents courses of lectures given at different times during the years 1913–1923 by Professor Whittaker to undergraduate and graduate students in the Mathematical Laboratory of the University of Edinburgh, and may be regarded as a manual of the teaching and practice of the Laboratory, complete save for the subject of Descriptive Geometry, for which a separate text-book is desirable. The manuscript of the lectures has been prepared for publication by Mr. Robinson, who has performed the whole of the work of numerical verification and has contributed additional examples.

The exposition has been designed to make each chapter, as far as possible, intelligible to those who have not mastered the preceding chapters; so that any one who is interested in some special problem may, by the help of the Table of Contents, find and understand what is said about it without being obliged to read from the beginning of the book.

One feature which perhaps calls for a word of explanation is the prominence given to arithmetical, as distinguished from graphical, methods. When the Edinburgh Laboratory was established in 1913, a trial was made, as far as possible, of every method which had been proposed for the solution of the problems under consideration, and many of these methods were graphical. During the ten years which have elapsed since then, the graphical methods have almost all been abandoned, as their inferiority has become evident, and at the present time the work of the Laboratory is almost exclusively arithmetical. A rough sketch on squared paper is often useful, but (except in Descriptive Geometry) graphical work performed carefully with instruments on a drawing-board is generally less rapid and less accurate than the arithmetical solution of the same problem.

The material equipment essential for a student's mathematical laboratory is very simple. Each student should have a copy of Barlow's tables of squares, etc., a copy of Crelle's "Calculating Tables," and a seven-place table of logarithms. Further, it is necessary to provide a stock of computing paper (*i.e.* paper ruled into squares by rulings a quarter of an inch apart; each square is intended to hold two digits; the rulings should be very faint, so as not to catch the eye more than is necessary to guide the alignment of the calculation), and lastly, a stock of computing forms for practical Fourier analysis (those used in Chapter X. of this book may be purchased). With this modest apparatus nearly all the computations hereafter described may be performed, although time and labour may often be saved by the use of multiplying and adding machines when these are available.

Attention may be drawn to the opportunities which the subject presents to the research worker in Mathematics. There is an evident need for new and improved methods of dealing with many of the problems discussed in the later chapters.

The authors wish to acknowledge their indebtedness to Mr. G. J. Lidstone, F.I.A., F.R.S.E., whose unique knowledge of actuarial literature has been of constant value to them. To him mainly it is due that the book is so rich in theorems and methods derived from actuarial sources, with which mathematicians are not usually well acquainted. Mr. Lidstone and Dr. John Dougall, F.R.S.E., have read the proof-sheets; by their suggestions many references have been added, many demonstrations simplified, and many obscurities removed.

E. T. W.
G. R.

PREFACE TO THE SECOND EDITION

Many correspondents have kindly sent notes of errata, which have been corrected in the present impression.

I gratefully acknowledge the help of Dr. A. C. Aitken, Lecturer in Mathematics and Statistics in the University of Edinburgh, whose recent contributions to the theory of Graduation have made it necessary to rewrite the latter part of Chapter XI.

E. T. W.

OCTOBER, 1925.

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CHAPTER I

INTERPOLATION WITH EQUAL INTERVALS OF THE ARGUMENT

1. Introduction.—Mathematics is occupied largely with the idea of *correspondence*: *e.g.* to every number x there corresponds a value of x^2, thus

$$x = 1,\ 2,\ 3,\ 4,\ 5,\ \dots$$
$$x^2 = 1,\ 4,\ 9,\ 16,\ 25,\ \dots$$

One of the two variables between which correspondence holds is called the *argument* and the other is called the *function* of that argument.

If a function y of an argument x is defined by an equation $y = f(x)$, where $f(x)$ is an algebraical expression involving only arithmetical operations such as squaring, dividing, etc., then by performing these operations we can find accurately the value of y, which corresponds to any value of x. But if $y = \log_{10} x$ (say), it is not possible to calculate y by performing simple arithmetical operations on x (at any rate it is not possible to calculate y accurately by performing a finite number of such operations), and we are compelled to have recourse to a *table*, which gives the values of y corresponding to certain selected values of x; *e.g.*

x.	$\log x$.	x.	$\log x$.
7·0	0·845 098	7·4	0·869 232
7·1	0·851 258	7·5	0·875 061
7·2	0·857 332	7·6	0·880 814
7·3	0·863 323	7·7	0·886 491

The question then arises as to how we can find the values of the function $\log x$ for values of the argument x which are intermediate between the tabulated values, *e.g.* such a value as $x=7{\cdot}152$. The answer to this question is furnished by the theory of *Interpolation*, which in its most elementary aspect may be described as the science of "reading between the lines of a mathematical table."

In the further development of the theory of interpolation it will be shown how to find the differential coefficient of a function which is specified by a table, and also to find its integral taken between any bounds of integration.

A kind of interpolation was used by Briggs,* but interpolation of the kind hereafter explained, based on the representation of functions by polynomials, was first introduced by James Gregory † in 1670.

2. Difference Tables.—Suppose a function $f(u)$ is given in a table for the values a, $a+w$, $a+2w$, $a+3w$, . . . of its argument u. It is required to find the value of the function when the argument has the value $a+xw$, where x is a fraction.

Before this problem can be solved by the method of interpolation, it is first necessary to form what are called the *differences* of the tabular values. The quantity

$$f(a+w)-f(a)$$

is denoted by $\Delta f(a)$ and is called the *first difference* of $f(a)$. The first difference of $f(a+w)$ is $f(a+2w)-f(a+w)$, which is denoted by $\Delta f(a+w)$. Moreover, the quantity

$$\Delta f(a+w)-\Delta f(a)$$

is denoted by $\Delta^2 f(a)$ and is called the *second difference* of $f(a)$, while the quantity

$$\Delta^2 f(a+w)-\Delta^2 f(a)$$

is denoted by $\Delta^3 f(a)$ and is called the *third difference* of $f(a)$, and so on.

It is convenient to arrange the tabular values and their differences for increasing values of the argument in what is called a *difference table*, as follows:

---

* Briggs' method was, however, closely related to the modern central-difference formulae. Cf. his *Arithmetica Logarithmica*, ch. xiii., and his *Trigonometria Britannica*, ch. xii. Cf. *Journal of the Institute of Actuaries*, **14**, pp. 1, 73, 84, 88; **15**, p. 312.

† Rigaud's *Correspondence of Scientific Men of the 17th Century*, **2**, p. 209.

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
a	$f(a)$			
		$\Delta f(a)$		
$a+w$	$f(a+w)$		$\Delta^2 f(a)$	
		$\Delta f(a+w)$		$\Delta^3 f(a)$
$a+2w$	$f(a+2w)$		$\Delta^2 f(a+w)$	
		$\Delta f(a+2w)$		$\Delta^3 f(a+w)$
$a+3w$	$f(a+3w)$		$\Delta^2 f(a+2w)$	
		$\Delta f(a+3w)$		$\Delta^3 f(a+2w)$
$a+4w$	$f(a+4w)$		$\Delta^2 f(a+3w)$	

and similarly for differences of order higher than the third. The first entry $f(a)$ is called the *leading term*, and the differences of $f(a)$, that is to say $\Delta f(a)$, $\Delta^2 f(a)$, . . . are called the *leading differences*. Evidently *each difference in the table is the number* (*with its proper algebraic sign*) *obtained by subtracting the number immediately above and to the left from the number immediately below and to the left.*

The sum of the entries in any column of differences is equal to the difference between the first and last entries of the preceding column. This affords a numerical check on the accuracy of the table. Thus in the above table we have

$$\Delta^2 f(a+3w) = \Delta^2 f(a) + \Delta^3 f(a) + \Delta^3 f(a+w) + \Delta^3 f(a+2w).$$

An example of a difference table is the following, which represents the natural sines of angles from 25° 40′ 0″ to 25° 43′ 0″ inclusive at intervals of 20″.

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
25° 40′ 0″	0·43313 47858 66963			
		8 73933 05476		
20″	0·43322 21791 72439		− 40 73056	
		8 73892 32420		− 822
40″	0·43330 95684 04859		− 40 73878	
		8 73851 58542		− 822
25° 41′ 0″	0·43339 69535 63401		− 40 74700	
		8 73810 83842		− 820
20″	0·43348 43346 47243		− 40 75520	
		8 73770 08322		− 823
40″	0·43357 17116 55565		− 40 76343	
		8 73729 31979		− 821
25° 42′ 0″	0·43365 90845 87544		− 40 77164	
		8 73688 54815		− 821
20″	0·43374 64534 42359		− 40 77985	
		8 73647 76830		− 822
40″	0·43383 38182 19189		− 40 78807	
		8 73606 98023		
25° 43′ 0″	0·43392 11789 17212			

It will be seen that in this case the third differences are practically constant when quantities beyond the fifteenth place are neglected, any departure from constancy in the last place being really due to the neglect of the sixteenth place of decimals in the original entries. So the fourth differences are zero.

It will be found that *in the case of practically all tabular functions the differences of a certain order are all zero; or, to speak more accurately, they are smaller than one unit in the last decimal place retained in the tables in question.* This fact lies at the basis of the method of interpolation, as we shall now see.

3. Symbolic Operators.—The formulae of the calculus of differences may be very simply represented by the use of what are called *symbolic operators.* Of these we have already introduced Δ, and we shall now consider another operator denoted by E.

Let w represent the interval between successive values of the argument of the function $f(a)$, and let E denote the operation of increasing the argument by w, so that $\mathrm{E}f(a)=f(a+w)$; in general we shall write $\mathrm{E}^x f(a)=f(a+xw)$, where x is an integer. Now by definition we had $\Delta f(a+xw)=f(a+xw+w)-f(a+xw)$, so $\Delta f(a+xw)=(\mathrm{E}-1)f(a+xw)$. It is therefore evident that the operators E and Δ are connected by the relation $\Delta=\mathrm{E}-1$ or

$$\mathrm{E}=1+\Delta.$$

When symbolic operators obey the ordinary laws of Algebra they may be separated from the symbols representing the functions to which they refer and treated independently in much the same way as symbols of quantity. Now it may be easily shown that the following relations are true for the operator Δ:

$$\Delta\{f(a)+f(b)+f(c)+\ldots\}=\Delta f(a)+\Delta f(b)+\Delta f(c)+\ldots,$$

$$\Delta kf(a)=k\Delta f(a), \text{ where } k \text{ is a constant factor,}$$

$$\Delta^m\Delta^n f(a)=\Delta^{m+n}f(a), \text{ where } m, n \text{ are positive integers.}$$

The corresponding identities for E are:

$$\mathrm{E}\{f(a)+f(b)+f(c)+\ldots\}=\mathrm{E}f(a)+\mathrm{E}f(b)+\mathrm{E}f(c)+\ldots,$$

$$\mathrm{E}kf(a)=k\mathrm{E}f(a),$$

$$\mathrm{E}^m\mathrm{E}^n f(a)=\mathrm{E}^{m+n}f(a).$$

Thus in many respects the operators E and Δ behave like algebraic symbols and may be combined like them.

The following examples illustrate the use of these operators:

Ex. 1.—*To express the nth differences of a tabulated function in terms of the successive entries.*

$$\begin{aligned}\Delta^n f(a) &= (\mathrm{E}-1)^n f(a)\\ &= \left\{\mathrm{E}^n - n\mathrm{E}^{n-1} + \frac{n(n-1)}{2!}\mathrm{E}^{n-2} - \ldots + (-1)^n\right\} f(a),\end{aligned}$$

i.e.

$$\Delta^n f(a) = f(a+nw) - nf(a+nw-w) + \frac{n(n-1)}{2!}f(a+nw-2w) - \ldots + (-1)^n f(a).$$

Ex. 2.—*To express the function $f(a+xw)$ in terms of $f(a)$ and the successive differences of $f(a)$, when x is a positive integer.*

$$\begin{aligned}f(a+xw) &= \mathrm{E}^x f(a)\\ &= (1+\Delta)^x f(a),\end{aligned}$$

so that

$$f(a+xw) = f(a) + x\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \ldots + \Delta^x f(a).$$

4. The Differences of a Polynomial.—We find without difficulty that the difference table for the function $y=x^3$ is as follows:

x.	y.	Δ.	Δ^2.	Δ^3.	Δ^4.
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	0
6	216	~~126~~ 126	36		

It will be seen that the third differences of this function are rigorously constant and the fourth differences are zero. This is a particular case of a general property which we shall now establish.

Note that the table may be extended indefinitely when we know the third differences to be constant. For by definition, when we add to an entry in a column of differences the corresponding first difference, the sum so formed gives the next entry in the column. It follows that the column of second differences can be formed from the leading term 6 by repeatedly adding the constant third difference 6; the column of first differences being formed from the leading term 1 by adding in succession the second differences 6, 12, 18, . . . The values of x^3 are then obtained from the leading term 0 by adding in succession the first differences 1, 7, 19, 37, 61, . . .

Consider the case when the tabulated function $f(x)$ is a polynomial of degree n, say,

$$f(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Lx + M.$$

Then

$$\begin{aligned}\Delta f(a) &= f(a+w) - f(a)\\ &= A\{(a+w)^n - a^n\} + B\{(a+w)^{n-1} - a^{n-1}\} + \ldots + Lw.\end{aligned}$$

Now

$$(a+w)^n = a^n + nwa^{n-1} + \frac{n(n-1)}{2!}w^2a^{n-2} + \ldots + w^n,$$

so that

$$\begin{aligned}\Delta f(a) = A\{nwa^{n-1} &+ \frac{n(n-1)}{2!}w^2a^{n-2} + \ldots + w^n\}\\ &+ B\{(n-1)wa^{n-2} + \frac{(n-1)(n-2)}{2!}w^2a^{n-3} + \ldots + w^{n-1}\}\\ &+ \ldots\\ &+ Lw.\end{aligned}$$

This is a polynomial of degree $(n-1)$ in a, and therefore the first differences of a polynomial represent another polynomial of degree less by one unit.

By repeated application of this result we see that
the 2nd differences represent a polynomial of degree $n-2$,
„ 3rd „ „ „ „ $n-3$,
. . . .
„ nth „ „ „ „ 0,

i.e. the nth differences are constant. It follows, therefore, that *the $(n+1)$th differences of a polynomial of the nth degree are all zero.*

5. The Differences of Zero.—A table of values of any power of the natural numbers may be formed by simple addition when the leading term and the leading differences are known, in precisely the same way as in forming the table of cubes (§ 4). The differences of the leading term 0^p, which are generally used in forming a table of x^p, are known as *the differences of zero.* They are of frequent occurrence in the calculus of differences.

In order to form a table of reference of the differences of zero we apply the result of § 3 (Ex. 1),

$$\Delta^n f(a) = f(a+nw) - nf(a+nw-w) + \tfrac{1}{2}n(n-1)f(a+nw-2w) - \ldots$$

and write

$$\Delta^n x^p = (x+n)^p - n\{x+(n-1)\}^p + \tfrac{1}{2}n(n-1)\{x+(n-2)\}^p - \ldots$$

If we now substitute in this equation particular values for x, p, and n, we obtain the equations

$$\Delta^n 0^p = n^p - n(n-1)^p + \tfrac{1}{2}n(n-1)(n-2)^p - \ldots \pm n.1^p \mp 0^p,$$
$$\Delta^{n-1}1^{p-1} = n^{p-1} - (n-1)^p + \tfrac{1}{2}(n-1)(n-2)^p - \ldots \pm 1^{p-1},$$

and therefore
$$\Delta^n 0^p = n\Delta^{n-1}1^{p-1}. \tag{1}$$

From the relation $\Delta^{n-1}f(a+w) = \Delta^n f(a) + \Delta^{n-1}f(a)$ we see that $\Delta^{n-1}1^{p-1} = \Delta^n 0^{p-1} + \Delta^{n-1}0^{p-1}$, and equation (1) may be written

$$\Delta^n 0^p = n(\Delta^n 0^{p-1} + \Delta^{n-1}0^{p-1}). \tag{2}$$

We now construct a table of values of $\Delta^n 0^p$ by the repeated application of this equation, remembering that $\Delta^0 0^1 = 0$, $\Delta^1 0^1 = 1$, and also that $\Delta^n 0^p = 0$ for $n > p$.

p.	$\Delta 0^p$.	$\Delta^2 0^p$.	$\Delta^3 0^p$.	$\Delta^4 0^p$.	$\Delta^5 0^p$.	$\Delta^6 0^p$.
1	1					
2	1	2				
3	1	6	6			
4	1	14	36	24		
5	1	30	150	240	120	
6	1	62	540	1560	1800	720
7	1	126	1806	8400	16800	15120
8	1	254	5796	40824	126000	191520
9	1	510	18150	186480	834120	1905120
10	1	1022	55980	818520	5103000	16435440

From equation (2) we see that the value of a particular difference $\Delta^n 0^p$ is obtained by taking n times the sum of the two numbers of the preceding row which are situated in the same column and in the preceding column respectively. For example,

$$\Delta^3 0^7 = 3(62+540)$$
$$= 1806.$$

6. The Differences of $x(x-1)(x-2)\ldots(x-p+1)$.—Among the polynomials of degree p there is one polynomial of special interest in the theory of interpolation, namely,

$$x(x-1)(x-2)\ldots(x-p+1).$$

This polynomial is denoted by $[x]^p$ and is called a *factorial*. If we suppose the interval of the argument in the difference table of $[x]^p$ to be unity, we have

$$
\begin{aligned}
[a]^p &= a(a-1)(a-2)\ \ldots\ (a-p+1),\\
[a+1]^p &= (a+1)a(a-1)(a-2)\ \ldots\ (a-p+2),\\
\Delta[a]^p &= [a+1]^p - [a]^p\\
&= a(a-1)(a-2)(a-3)\ldots(a-p+2)\{(a+1)-(a-p+1)\}\\
&= p[a]^{p-1},
\end{aligned}
$$

so that $$\Delta[x]^p = p[x]^{p-1}.\text{*}$$

It follows that

$$\frac{\Delta[x]^p}{p!} = \frac{[x]^{p-1}}{(p-1)!}, \text{ or } \frac{[x+1]^p}{p!} = \frac{[x]^p}{p!} + \frac{[x]^{p-1}}{(p-1)!},$$

a result that may now be used to tabulate the values of $[x]^p/p!$ as in the following table:

x.	$[x]^2/2!$.	$[x]^3/3!$.	$[x]^4/4!$.	$[x]^5/5!$.
0				
1	0			
2	1	0		
3	3	1	0	
4	6	4	1	0
5	10	10	5	1
6	15	20	15	6
7	21	35	35	21
8	28	56	70	56
9	36	84	126	126

7. The Representation of a Polynomial by Factorials.—In § 4 we found an expression for $\Delta f(x)$, the first difference of a polynomial of degree n, in a form which is less simple than the polynomial itself. It is more convenient to carry out the operation of differencing by the use of factorials, using the relation of § 6:

$$\Delta[x]^p = p[x]^{p-1}. \qquad (1)$$

Let $\phi_k(x)$ denote a polynomial in x of degree k. We may write $\phi_k(x) = r + (x-n+k)\phi_{k-1}(x)$, where r is the remainder and $\phi_{k-1}(x)$ the quotient when $\phi_k(x)$ is divided by $(x-n+k)$, so $\phi_{k-1}(x)$ is of degree $(k-1)$. By a repeated application of this transformation, we obtain an expression for a polynomial of the nth degree in terms of factorials:

---

* This is analogous to the formula of the differential calculus $\frac{d}{dx}(x^p) = px^{p-1}$.

$$\begin{aligned}\phi_n(x) &= \alpha + [x]\phi_{n-1}(x)\\ &= \alpha + \beta[x] + [x]^2\phi_{n-2}(x)\\ &= \alpha + \beta[x] + \gamma[x]^2 + [x]^3\phi_{n-3}(x)\\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\ &= \alpha + \beta[x] + \gamma[x]^2 + . \; . \; . + [x]^n\phi_0(x),\end{aligned}$$

where α, β, γ, . . . are constants and $\phi_0(x)$ is a constant ν (say). We thus obtain the result

$$\phi_n(x) = \alpha + \beta[x] + \gamma[x]^2 + \delta[x]^3 + . \; . \; . + \nu[x]^n. \qquad (2)$$

Ex.—To represent the function $y = x^4 - 12x^3 + 42x^2 - 30x + 9$ and its successive differences in the factorial notation.

Using detached coefficients when dividing by x, $x - 1$, $x - 2$, . . .,[*]

$$\begin{array}{c|rrrrl}
1 & 1 & -12 & +42 & -30 & \multicolumn{1}{|l}{9} \\
 & 0 & +1 & -11 & +31 & \multicolumn{1}{|l}{} \\ \cline{2-5}
2 & 1 & -11 & +31 & \multicolumn{1}{|l}{1} & \\
 & 0 & +2 & -18 & \multicolumn{1}{|l}{} & \\ \cline{2-4}
3 & 1 & -9 & \multicolumn{1}{|l}{13} & & \\
 & 0 & +3 & \multicolumn{1}{|l}{} & & \\ \cline{2-3}
\underline{1} & -6 & & & &
\end{array}$$

we obtain the value of y in the form

$$y = [x]^4 - 6[x]^3 + 13[x]^2 + [x] + 9.$$

The successive differences are given by

$$\begin{aligned}\Delta y &= 4[x]^3 - 18[x]^2 + 26[x] + 1,\\ \Delta^2 y &= 12[x]^2 - 36[x] + 26,\\ \Delta^3 y &= 24[x] - 36,\\ \Delta^4 y &= 24.\end{aligned}$$

Now let a be one of the tabulated values of the argument of a polynomial of degree n, and let w be the interval between successive values of the argument. Consider the value $f(a+xw)$ of the polynomial corresponding to the value $(a+xw)$ of the argument. Writing $f(a+xw)$ for $\phi_n(x)$ in (2) and applying the operation denoted by equation (1) to both sides of equation (2), we find that

$$\Delta f(a+xw) = \beta + 2\gamma[x]^1 + 3\delta[x]^2 + . \; . \; . + n\nu[x]^{n-1}. \qquad (3)$$

---

[*] Chrystal, *Algebra*, **1**, p. 108.

Differencing this equation, we obtain

$$\Delta^2 f(a+xw) = 2\gamma + 2.3\delta[x]^1 + 3.4\epsilon[x]^2 + \ldots + n(n-1)\nu[x]^{n-2}. \quad (4)$$

Moreover,

$$\Delta^3 f(a+xw) = 2.3.\delta + 2.3.4\epsilon[x]^1 + 3.4.5\xi[x]^2 + \ldots$$
$$+ n(n-1)(n-2)\nu[x]^{n-3}, \quad (5)$$

and so on for differences of higher order. The values of the coefficients α, β, γ, . . . are found by putting $x=0$ in each of the equations (2), (3), (4), . . . so that

$$\alpha = f(a), \quad \beta = \Delta f(a), \quad \gamma = \tfrac{1}{2}\Delta^2 f(a), \quad \delta = \tfrac{1}{6}\Delta^3 f(a), \ldots \quad \nu = \Delta^n f(a)/n!.$$

Equation (2) may now be written

$$f(a+xw) = f(a) + x\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \ldots$$
$$+ \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}\Delta^n f(a).$$

This formula enables us to express the polynomial* $f(a+xw)$ *in terms of the factorials* x, $x(x-1)$, $x(x-1)(x-2)$, . . . *when a difference table of the function is given.*

This general formula may be easily verified for special values of x.
When $x=0$, it becomes $f(a) = f(a)$.
When $x=1$, then

$$f(a+w) = f(a) + 1.\Delta f(a)$$
$$= f(a) + \{f(a+w) - f(a)\}, \text{ which is an identity.}$$

When $x=2$,

$$f(a+2w) = f(a) + 2\Delta f(a) + \Delta^2 f(a)$$
$$= f(a) + 2\{f(a+w) - f(a)\}$$
$$+ \{f(a+2w) - 2f(a+w) + f(a)\}.$$

8. The Gregory-Newton Formula of Interpolation.—The general formula of the last section may be applied to solve the problem of interpolation.

Suppose that y is a function of an argument u and that the values of y given in the table are $f(a)$, $f(a+w)$, $f(a+2w)$, $f(a+3w)$, . . . corresponding to the values a, $a+w$, $a+2w$, $a+3w$, . . . of u. Also suppose that these values of the function are entered in a difference table and that the differences of order n are constant. We are just not supposed to know the values of y which correspond to other values of u, such as $u = a + \frac{1}{2}w$.

---

* Cf. Ex. 2, § 3.

It is required to find an analytical expression for these intermediate values of y.

The problem may be stated graphically as follows:

Draw the rectangular axes Ou, Oy. Let $K, L, M, N \ldots$ be points on the u axis having abscissae a, $a+w$, $a+2w$, $a+3w$, ... respectively. At these points erect ordinates KA, LB, MC, ND, ... equal respectively to the entries $f(a)$, $f(a+w)$, $f(a+2w)$, $f(a+3w)$, ... Then the points $A, B, C, D, \ldots$ so determined are points on the graph of the function.* The problem of finding a "smooth" curve to pass through the points $A, B, C, D, \ldots$ has not a unique solution: in fact an infinite number of curves satisfying these conditions can be found. As our aim is a practical one, we naturally choose the simplest solution of our problem. Remembering that the simplest functions are polynomials, we inquire if it is possible to pass through the points $A, B, C, \ldots$ a curve which is the graph of a *polynomial* function of degree n.

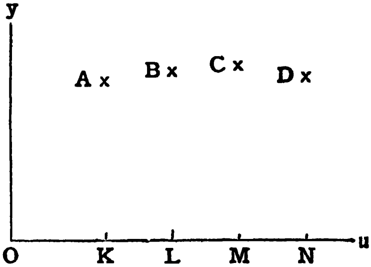

FIG. 1.

We have already seen (§ 4) that for any polynomial of degree n the differences of order n are constant and for the set of values $f(a), f(a+w), f(a+2w), \ldots$ it has been assumed that the differences of order n are constant. This being so, a polynomial of degree n exists which takes the values $f(a)$, $f(a+w)$, $f(a+2w)$, ... when the argument u has the values a, $a+w$, $a+2w$, ...; in fact, by the last section, we can write down an expression for the polynomial. It is

$$y=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots$$

$$+\frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n f(a) \quad (1)$$

$$=f(u),$$

---

* We do not know anything about the portions of the graph intermediate between these points, but we assume that the graph is a *smooth curve*; for our present purpose we can take this to mean that the function has finite differential coefficients of all orders at every point.

where x is connected with u by the relation $u=a+xw$, and where

$\Delta f(a)$ stands for $f(a+w)-f(a)$,
$\Delta^2 f(a)$ stands for $f(a+2w)-2f(a+w)+f(a)$,
and so on.

We shall now take the polynomial (1) *to represent the function y also for values of the argument intermediate between the tabulated values.* The portions of the graph intermediate between the points $A, B, C, \ldots$ may therefore be filled in by drawing the curve

$$\begin{aligned} y &= f(a+xw) \\ &= f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots \end{aligned} \qquad (2)$$

and in order to compute the value of y corresponding to any intermediate value of the argument such as $a+\frac{1}{2}w$, we simply substitute the value $x=\frac{1}{2}$ in this formula,* which is the analytical expression required.

The fundamental problem of interpolation is thus solved. The formula (1) is often referred to as *Newton's formula of interpolation,* although it was discovered by James Gregory in 1670.†

The application of the Gregory-Newton formula is illustrated by the following examples:

---

* Many books of logarithmic tables, etc., contain a table of the binomial coefficients required in the interpolation formula (1), at intervals of 0·01 from $x=0$ to $x=1$.

† Cf. a letter of Gregory to Collins of date November 23, 1670, printed in Rigaud's *Correspondence*, **2**, p. 209. An example of the use of the formula is worked out on p. 211 of Rigaud. Collins was accustomed to send on to Newton the mathematical discoveries of Gregory (cf. Rigaud, **2**, p. 335).

Newton's publications on interpolation are contained in:

1. The *Methodus Differentialis* published in 1711 but written before October 1676.
2. A letter written in 1676 to John Smith.
3. Lemma v. in Book iii. of the *Principia* published in 1687. The above formula is Case i.
4. Various references in the *Commercium Epistolicum* of dates 1672/3 to 1676.

These have been collected and edited by D. C. Fraser in the *Journal of the Institute of Actuaries*, **51** (1918–19), pp. 77 and 211.

Ex. 1.—*From the table given below to find the entry corresponding to* $x = 21$.

Argument.	*Entry.*	Δ.	Δ².	Δ³.	Δ⁴.
20	0·229314955248				
		701747247			
22	0·230016702495		602297		
		702349544		− 1944	
24	0·230719052039		600353		4
		702949897		− 1940	
26	0·231422001936		598413		3
		703548310		− 1937	
28	0·232125550246		596476		
		704144786			
30	0·232829695032				

Here $a = 20$, $w = 2$, $f(a + xw) = f(21)$, and $x = \frac{1}{2}$.

$$f(21) = f(20) + x\Delta f(20) + \frac{x(x-1)}{2}\Delta^2 f(20) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(20) + \ldots$$

$$\begin{aligned} &= 229314955248 + \tfrac{1}{2}(701747247) - \tfrac{1}{8}(602297) - \tfrac{1}{16}(1944) \\ &= 229314955248 \qquad - \begin{cases} 75287\cdot 1 \\ +\ 121\cdot 5 \end{cases} \\ &\quad + 350873623\cdot 5 \\ &= 229665828871\cdot 5 - 75408\cdot 6 \end{aligned}$$

so

$$f(21) = 0\cdot 229665753463.$$

Ex. 2.—*To find the co-ordinate* X *of the sun on November* 10, 1910, *at* $4^h\ 30^m$ *G.M.T.* (X *is the sun's true geocentric co-ordinate measured on a line passing through the true equinox of the date*).

The *Nautical Almanac* gives the following readings from which we construct a difference table:

1910.	−X.	Δ.	Δ².	Δ³.
November 9·0	0·6850997			
		− 63809		
9·5	0·6787188		− 514	
		− 64323		4
10·0	0·6722865		− 510	
		− 64833		7
10·5	0·6658032		− 503	
		− 65336		2
11·0	0·6592696		− 501	
		− 65837		
11·5	0·6526859			

We must interpolate for $4^h\ 30^m$ from November 10·0. The interval is 12^h. Then $4^h\ 30^m$, as a fraction of the argument, gives $x = 0\cdot 375$.

$$\begin{aligned}\log x &= 9{\cdot}5740313\\ \log(x-1) &= 9{\cdot}7958800(n),\end{aligned}$$

where (n) indicates that 9·7958800 is the logarithm of a negative number

$$\begin{aligned}\log \tfrac{1}{2} &= 9{\cdot}6989700\\ \log \tfrac{1}{2}x(x-1) &= 9{\cdot}0688813(n)\\ \log \tfrac{1}{3} &= 9{\cdot}5228787\\ \log(x-2) &= 0{\cdot}2108534(n)\\ \log \tfrac{1}{6}x(x-1)(x-2) &= 8{\cdot}8026134.\end{aligned}$$

Also

$$\begin{aligned}\log(-64833) &= 4{\cdot}8117961(n)\\ \log x &= 9{\cdot}5740313\\ \log(-64833x) &= 4{\cdot}3858274(n)\\ &= \log(-24312{\cdot}4)\end{aligned}$$

$$\begin{aligned}\log(-503) &= 2{\cdot}7015680(n)\\ \log \tfrac{1}{2}x(x-1) &= 9{\cdot}0688813(n)\\ \log \tfrac{1}{2}x(x-1)(-503) &= 1{\cdot}7704493\\ &= \log 58{\cdot}94\end{aligned}$$

$$\begin{aligned}\log 2 &= 0{\cdot}3010300\\ \log \tfrac{1}{6}x(x-1)(x-2) &= 8{\cdot}8026134\\ \log \tfrac{1}{6}x(x-1)(x-2)(2) &= 9{\cdot}1036434 = \log 0{\cdot}1.\end{aligned}$$

Therefore $\quad -\mathrm{X} = 0{\cdot}67228650 - 0{\cdot}00243124 + 0{\cdot}00000589,$

and finally $\quad -\mathrm{X} = 0{\cdot}6698612.$

9. An Alternative Form of the Gregory-Newton Formula.—The Gregory-Newton formula may be written in an alternative form which is convenient when an arithmometer * is used. Rearranging the formula of the last section in the form

$$f(a+xw) = f(a) + x[\Delta f(a) - \tfrac{1}{2}(1-x)\{\Delta^2 f(a) - \tfrac{1}{3}(2-x)(\ .\ .\ .\ )\}],$$

and assuming the differences of order n to be constant, we may replace the Gregory-Newton formula by

$$f(a+xw) = f(a) + xu_1, \tag{1}$$

where

$$\begin{aligned}u_1 &= \Delta f(a) - \tfrac{1}{2}(1-x)u_2,\\ &\cdot\qquad\cdot\qquad\cdot\qquad\cdot\\ u_p &= \Delta^p f(a) - \frac{1}{p+1}(p-x)u_{p+1},\\ &\cdot\qquad\cdot\qquad\cdot\qquad\cdot\\ u_n &= \Delta^n f(a), \text{ which is constant.}\end{aligned}$$

When computing a value of the function by this method, we begin with the constant difference u_n and calculate in succession the values of $u_{n-1}, u_{n-2}, \ldots, u_1$, finally substituting the value of u_1 in equation (1). The following example will serve as an illustration of this method:

---

* When an arithmometer is not available *Crelle's Calculating Tables* will be found useful for this purpose.

Ex.—To find $f(\theta)$ *when* $\theta = 24^\circ\!\cdot 46980\ 05207\ 020$, *having given*

θ.	$f(\theta)$.	Δ.	Δ^2.	Δ^3.	Δ^4.
24·4	0·216 198 561 343				
		168 272 307			
24·5	0·216 366 833 650		745 715		
		169 018 022		768	
24·6	0·216 535 851 672		746 483		5
		169 764 505		773	
24·7	0·216 705 616 177		747 256		4
		170 511 761		777	
24·8	0·216 876 127 938		748 033		5
		171 259 794		782	
24·9	0·217 047 387 732		748 815		
		172 008 609			
25·0	0·217 219 396 341				

Here $w = 0\cdot 1,\quad a = 24^\circ\!\cdot 4,\quad x = 0\cdot 698\ 005\ 207\ 02.$

Hence $u_3 = \Delta^3 f(a) - \frac{1}{4}(3 - x)\Delta^4 f(a) = 768 - 0\cdot 576 \times 5$

$= 765\cdot 1,$

$u_2 = \Delta^2 f(a) - \frac{1}{3}(2 - x)u_3 = 745\ 715 - 0\cdot 434\ 0 \times 765\cdot 1$

$= 745\ 383\cdot 0,$

$u_1 = \Delta f(a) - \frac{1}{2}(1 - x)u_2 = 168\ 272\ 307 - 0\cdot 150\ 997\ 4 \times 745\ 383$

$= 168\ 159\ 756\cdot 1.$

Then

$f(a + xw) = f(a) + xu_1$

$= 0\cdot 216\ 198\ 561\ 343 + 0\cdot 698\ 005\ 207\ 02 \times 0\cdot 000\ 168\ 159\ 756$

$= 0\cdot 216\ 198\ 561\ 343$

$\quad + 117\ 376\ 385,$

or $\quad f(\theta) = 0\cdot 216\ 315\ 937\ 728.$

10. The Binomial Theorem.—By use of the operator **E**, we can write the Gregory-Newton interpolation formula in the form

$$E^x f(a) = \{1 + x\Delta + \frac{x(x-1)}{2!}\Delta^2 + \ldots\} f(a).$$

When thus written, the formula is seen to be the same as that obtained by expanding $(1 + \Delta)^x$ by the Binomial Theorem in ascending powers of Δ and then operating on $f(a)$ with the terms of the series so formed, *i.e.*

$$E^x f(a) = (1 + \Delta)^x f(a).$$

The Binomial Theorem was made known (in correspondence) six years after the Gregory-Newton formula; in fact, Newton seems to have discovered the Binomial Theorem by forming the expansions of $(1 + x)^n$ directly for integral values of n, and then writing down the powers of x in these expansions. In the case of the coefficient of x^2 he would have:

Exponent.	*Coefficient of x^2.*	Δ.	Δ^2.
0	0		
		0	
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		

whence evidently the coefficient is of the second degree in n. Since it vanishes when $n=0$ and also when $n=1$, it must contain the factors n and $(n-1)$; and, since the coefficient has the value 1 when $n=2$, it is $\frac{n(n-1)}{2}$.

We may remark that if we form a difference table for $(1+x)^n$ thus:

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
0	1			
		x		
1	$(1+x)^1$		x^2	
		$x(1+x)$		x^3
2	$(1+x)^2$		$x^2(1+x)$	
		$x(1+x)^2$		$x^3(1+x)$
3	$(1+x)^3$		$x^2(1+x)^2$	

then on substituting the values $f(0)=1$, $\Delta f(0)=x$ . . . in the Gregory-Newton formula

$$f(n)=f(0)+n\Delta f(0)+\tfrac{1}{2}n(n-1)\Delta^2 f(0)+\ldots$$

we obtain

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

which is the binomial expansion.

EXAMPLES ON CHAPTER I

1. Form the difference tables corresponding to the following entries:

	θ.	*log tan θ.*
(*a*)	26° 10′ 0″	9·691 380 858 103 01
	10″	434 054 052 28
	20″	487 246 020 72
	30″	540 434 009 42
	40″	593 618 019 47
	50″	646 798 051 97
	26° 11′ 0″	9·691 699 974 108 01
	10″	753 146 188 70
	20″	806 314 295 11
	30″	859 478 428 36

	x.	sin x.
(b)	28° 40′ 00″	0·479 713 113 250 246
	10″	755 651 470 168
	20″	798 188 562 452
	30″	840 724 526 998
	40″	883 259 363 705
	50″	925 793 072 474
	28° 41′ 00″	968 325 653 205
	10″	0·480 010 857 105 798

2. If $y = 2x^3 - x^2 + 3x + 1$, calculate the values of y corresponding to $x = 0, 1, 2, 3, 4, 5$, and form the table of differences. Prove theoretically that the second difference is $12x + 10$ and verify this numerically.

3. Find the function whose first difference is the function

$$\alpha x^3 + \beta x^2 + \gamma x + \delta.$$

4. Find the successive differences of

(*a*) $1/x$, the interval being unity,
(*b*) $\cos nx$, the interval being w.

5. Express $f(x) = 3x^3 + x^2 + x + 1$ in the form

$$\alpha x(x-1)(x-2) + \beta x(x-1) + \gamma x + \delta$$

by comparing coefficients. Calculate the values of $f(x)$ for $x = 0, 1, 2, 3, 4, 5$, etc., and form a difference table. Verify the equation

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2\,!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(0).$$

6. Compute the third difference of $f(51)$ by the formula of § 3, Ex. 1, from the following table of entries :

x	51	52	53	54
$f(x)$	132651	140608	148877	157464

verifying the result by means of a difference table.

7. Given the table of values

x	− 3	− 2	− 1	0	1
y	16	7	4	1	− 8

find by means of the Gregory-Newton formula an expression for y as a function of x.

8. Construct a difference table having given

log 5·980 = 0·776 701 184 0
log 5·981 = 0·776 773 802 4
log 5·982 = 0·776 846 408 7
log 5·983 = 0·776 919 002 8
log 5·984 = 0·776 991 584 9

and determine log 5·9805.

9. Let p, q, r, s be successive entries in a table corresponding to equidistant arguments.

Show that when third differences are taken into account the entry corresponding to the argument half-way between the arguments of q and r is

$$\frac{q+r}{2}+\frac{(q+r)-(p+s)}{16}. \qquad \text{(De Morgan.)}$$

10. Let p, q, r, s be successive entries (corresponding to equidistant arguments) in a table. It is required to interpose 3 entries (corresponding to equidistant arguments) between q and r, using third differences. Show that this may be done as follows:

Between q and r interpose 3 arithmetical means A, B, and C; also between $3q-2p-s$ and $3r-2s-p$ interpose 3 means A′, B′, and C′. Then the 3 terms required are $A+\frac{1}{32}A'$, $B+\frac{1}{24}B'$, $C+\frac{1}{32}C'$.

(De Morgan.)

✓ 11. Determine log 6·0405, having given

log 6·040 = 0·7810369386
log 6·041 = 0·7811088357
log 6·042 = 0·7811807209
log 6·043 = 0·7812525942
log 6.044 = 0·7813244557

✓ 12. Using the method of § 9, find sin 24°·4698005207, having given the values

θ.	sin θ.
24·25	0·410718852614
24·50	0·414693242656
24·75	0·418659737537
25·00	0·422618261741
25·25	0·426568739902
25·50	0·430511096808

✓ 13. Given the values

x.	$f(x)$.
0	858·313740095
1	869·645772308
2	880·975826766
3	892·303904583
4	903·630006875

calculate $f(1·5)$ by the Gregory-Newton formula.

14. The values of a function corresponding to the values 1, 2, 3, 4, 5 of the argument are 0·198669, 0·237702, 0·276355, 0·314566, 0·352274 respectively. Calculate the values of the function when the argument has the values 1·25 and 1·75 respectively.

15. Using the difference table given in § 2, find the values of sin 25° 40′ 10″ and sin 25° 40′ 30″. Also verify the answers

sin 25° 40′ 50″ = 0·433 353 261 493 416,
sin 25° 41′ 10″ = 0·433 440 644 614 711,
sin 25° 41′ 30″ = 0·433 528 023 660 896,
sin 25° 41′ 50″ = 0·433 615 398 631 149,

obtained by taking x numerically less than unity in the formula of § 8.

16. Calculate log tan 24° 0′ 5″, given the values

log tan 24° 0′ 0″ = 9·648 583 137 400 95
log tan 24° 0′ 20″ = 9·648 696 457 723 08
log tan 24° 0′ 40″ = 9·648 809 758 267 66
log tan 24° 1′ 0″ = 9·648 923 039 045 83
log tan 24° 1′ 20″ = 9·649 036 300 068 75
log tan 24° 1′ 40″ = 9·649 149 541 347 57.

17. The following table gives the values of $I(x) = \int_x^\infty e^{-s^2} ds$:

x.	$I(x)$.
0·00	0·886 226 92
0·01	0·876 227 24
0·02	0·866 229 57
0·03	0·856 235 90
0·04	0·846 248 22
0·05	0·836 268 53

Calculate $I(x)$ for $x = 0{\cdot}025$ by interpolation and verify your result by use of the formula

$$I(0) - I(x) = x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \dots$$

CHAPTER II

INTERPOLATION WITH UNEQUAL INTERVALS OF THE ARGUMENT

11. Divided Differences.—We have so far assumed that the values of the argument proceed by equal steps; but with data derived from observation it is not always possible to complete a difference table in this way. For example, when astronomical observations are disturbed by clouds there are gaps in the records.

Consider the case in which the values of the argument, for which the function is known, are unequally spaced, and suppose that the values of $f(x)$ are known for $x=a_0$, $x=a_1$, $x=a_2$, . . ., $x=a_n$, where the intervals a_1-a_0, a_2-a_1, a_3-a_2, . . ., a_n-a_{n-1} need not be equal. In place of ordinary differences we now introduce what are known as *divided differences.*[*] Let us form in succession the quantities

$$\frac{f(a_1)-f(a_0)}{a_1-a_0}=f(a_1,a_0),\ \frac{f(a_2)-f(a_1)}{a_2-a_1}=f(a_2,a_1),\ \frac{f(a_3)-f(a_2)}{a_3-a_2}=f(a_3,a_2),$$

and so on. These are called *divided differences of the first order.* Moreover, let us form

$$\frac{f(a_2,a_1)-f(a_1,a_0)}{a_2-a_0}=f(a_2,a_1,a_0),\ \frac{f(a_3,a_2)-f(a_2,a_1)}{a_3-a_1}=f(a_3,a_2,a_1).$$

These are called *divided differences of the second order.* Also let

$$\{f(a_3,a_2,a_1)-f(a_2,a_1,a_0)\}/(a_3-a_0)=f(a_3,a_2,a_1,a_0).$$

This is called a *divided difference of the third order.* The divided differences of higher orders are formed in the same way, so that the order of a divided difference is less by unity than the number of arguments required for its definition.

---

[*] Divided differences might fairly be ascribed to Newton, Lemma v. The term was used first by De Morgan, *Diff. and Int. Calc.* (1842), p. 550, and afterwards by Oppermann, *Journ. Inst. Act.* 15 (1869), p. 146. Ampère, *Ann. de Gergonne*, 26 (1826), p. 329, used the name *interpolatory functions*.

Divided differences may be expressed more symmetrically as follows:

$$f(a_1, a_0) = \frac{f(a_0)}{a_0 - a_1} + \frac{f(a_1)}{a_1 - a_0},$$

$$\begin{aligned} f(a_2, a_1, a_0) &= \frac{1}{a_2 - a_0}\left\{\frac{f(a_2)}{a_2 - a_1} + \frac{f(a_1)}{a_1 - a_2}\right\} + \frac{1}{a_0 - a_2}\left\{\frac{f(a_1)}{a_1 - a_0} + \frac{f(a_0)}{a_0 - a_1}\right\} \\ &= \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)} + \frac{f(a_1)}{(a_1 - a_0)(a_1 - a_2)} + \frac{f(a_2)}{(a_2 - a_1)(a_2 - a_0)}, \end{aligned}$$

$$\begin{aligned} f(a_3, a_2, a_1, a_0) &= \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)} + \frac{f(a_1)}{(a_1 - a_0)(a_1 - a_2)(a_1 - a_3)} \\ &\quad + \frac{f(a_2)}{(a_2 - a_0)(a_2 - a_1)(a_2 - a_3)} + \frac{f(a_3)}{(a_3 - a_0)(a_3 - a_1)(a_3 - a_2)}. \end{aligned}$$

In general, as may easily be shown by induction, a divided difference of the pth order is a symmetric function of its arguments and is in fact the sum of $(p+1)$ *functions of the form*

$$\frac{f(a_r)}{\text{difference-product of } a_r \text{ with } a_0, a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p}$$

It is evident from this statement that when the arguments required to form a particular divided difference are arranged in a different order, the value of the divided difference remains unchanged, *e.g.*

$$f(a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0) = f(a_0, a_1, a_2, \ldots, a_{n-1}, a_n).$$

Divided differences are arranged in a *table of divided differences* as follows:

Argument.	*Entry.*	Δ^1	Δ^2	Δ^3
a_0	$f(a_0)$			
		$f(a_0, a_1)$		
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		$f(a_1, a_2, a_3, a_4)$
a_3	$f(a_3)$		$f(a_2, a_3, a_4)$	
		$f(a_3, a_4)$		$f(a_2, a_3, a_4, a_5)$
a_4	$f(a_4)$		$f(a_3, a_4, a_5)$	

The following may serve as an example of a table of divided differences:

x.	$f(x)$.	Δ	Δ^2	Δ^3
0	132651			
		8113		
2	148877		158	
		8587		1
3	157464		162	
		8911		1
4	166375		167	
		9579		1
7	195112		173	
		10444		
9	216000			

In this example the differences of the third order are constant. We shall now see under what circumstances a column of constant divided differences is obtained.

12. Theorems on Divided Differences.

I. *If a function $f(x)$ is numerically equal to the sum of two functions $g(x)$, $h(x)$, for a set of values of the argument x, then any divided difference of $f(x)$ formed from those values is equal to the sum of the corresponding divided differences of $g(x)$ and $h(x)$.*

For example,

$$f(a_1, a_0) = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = \frac{\{g(a_1) - g(a_0)\} + \{h(a_1) - h(a_0)\}}{a_1 - a_0}$$

$$= g(a_1, a_0) + h(a_1, a_0),$$

and similarly for differences of higher order.

II. *A divided difference of $cf(x)$, where c is a constant factor, is c times the corresponding divided difference of $f(x)$.*

For example, the divided difference of the first order of $cf(x)$ is

$$\frac{cf(a_1) - cf(a_0)}{a_1 - a_0} = c\frac{f(a_1) - f(a_0)}{a_1 - a_0} = cf(a_1, a_0).$$

III. *The divided differences of order n of x^n are constant (where n is a positive integer).*

Let $f(x) = x^n$.

Then $f(a_0, a_1) = (a_0^n - a_1^n)/(a_0 - a_1)$

$$= a_0^{n-1} + a_1 a_0^{n-2} + \ldots + a_1^{n-1}$$

a homogeneous function of a_0, a_1 of degree $(n-1)$. Moreover,

$$
\begin{aligned}
f(a_0, a_1, a_2) &= \frac{[a_0^{n-1} + a_1 a_0^{n-2} + \dots + a_1^{n-1}] - [a_2^{n-1} + a_1 a_2^{n-2} + \dots + a_1^{n-1}]}{a_0 - a_2} \\
&= (a_0^{n-1} - a_2^{n-1})/(a_0 - a_2) + a_1(a_0^{n-2} - a_2^{n-2})/(a_0 - a_2) + \dots \\
&\qquad + a_1^{n-2}(a_0 - a_2)/(a_0 - a_2) \\
&= (a_0^{n-2} + a_2 a_0^{n-3} + \dots + a_2^{n-2}) \\
&\qquad + a_1(a_0^{n-3} + a_2 a_0^{n-4} + \dots + a_2^{n-3}) + \dots
\end{aligned}
$$

which is a homogeneous function of a_0, a_1, a_2 of degree $(n-2)$. In general $f(a_0, a_1, a_2, \dots, a_p)$ is a homogeneous function of $a_0, a_1, a_2, \dots, a_p$ of degree $(n-p)$. Taking $p = n$, we see that $f(a_0, a_1, a_2, \dots, a_n)$ is a constant.

Corollary: The divided differences of order $(n+1)$ of x^n are zero.

IV. *The divided differences of order n of a polynomial of the nth degree are constant.*

This theorem follows immediately from theorems I., II., and III., since the divided difference of order n of each of the terms whose degree is less than n is zero.

V. *A divided difference of order r may be expressed as the quotient of two determinants each of order $r+1$.*

Consider the divided difference of the third order,

$$
\begin{aligned}
f(a_0, a_1, a_2, a_3) &= \Sigma \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)} \\
&= \Sigma \frac{f(a_0)\ (\text{difference-product of } a_1, a_2, a_3)}{\text{difference-product of } a_0, a_1, a_2, a_3}.
\end{aligned}
$$

Now a difference-product may be expressed as a determinant of the kind known as Vandermonde's, thus

$$
(\text{difference-product of } a_1, a_2, a_3) = \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1 & a_2 & a_3 \\ 1 & 1 & 1 \end{vmatrix}
$$

Therefore

$$
f(a_0, a_1, a_2, a_3) = \Sigma \frac{f(a_0) \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1 & a_2 & a_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}
$$

or

$$f(a_0, a_1, a_2, a_3) = \frac{\begin{vmatrix} f(a_0) & f(a_1) & f(a_2) & f(a_3) \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}$$

and so in general for differences of order higher than the third.

13. Newton's Formula for Unequal Intervals.—Let $f(u)$ be a function whose divided differences of (say) order 4 vanish or are negligible; and suppose its values for 4 arguments a_0, a_1, a_2, a_3 are known so that the table of divided differences is as follows:

Argument.	*Entry.*	Δ		
a_0	$f(a_0)$			
		$f(a_0, a_1)$		
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		$\underline{f(a_1, a_2, a_3, a_4)}$
a_3	$f(a_3)$			

We may obtain the value of the function for any other argument u in the following way. Beginning with the constant difference which is of order 3, we have

$$f(u, a_0, a_1, a_2) = f(a_0, a_1, a_2, a_3). \tag{1}$$

By definition of the divided difference of order 2,

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) + (u - a_2)f(u, a_0, a_1, a_2),$$

and therefore

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) + (u - a_2)f(a_0, a_1, a_2, a_3). \tag{2}$$

Again by definition,

$$f(u, a_0) = f(a_0, a_1) + (u - a_1)f(u, a_0, a_1), \tag{3}$$

and substituting in this equation the value of $f(u, a_0, a_1)$ from (2),

$$f(u, a_0) = f(a_0, a_1) + (u - a_1)f(a_0, a_1, a_2) + (u - a_1)(u - a_2)f(a_0, a_1, a_2, a_3).$$

Also by definition $\quad f(u) = f(a_0) + (u - a_0)f(u, a_0), \qquad (4)$

or

$$f(u) = f(a_0) + (u - a_0)f(a_0, a_1) + (u - a_0)(u - a_1)f(a_0, a_1, a_2) + (u - a_0)(u - a_1)(u - a_2)f(a_0, a_1, a_2, a_3). \tag{5}$$

From the equations (1), (2), (3), (4) the quantities $f(u, a_0, a_1, a_2)$, $f(u, a_0, a_1)$, $f(u, a_0)$, $f(u)$ are now known and may be inserted in the table of divided differences thus:*

Argument.	*Entry.*			
u	$f(u)$			
		$f(u, a_0)$		
a_0	$f(a_0)$		$f(u, a_0, a_1)$	
		$f(a_0, a_1)$		$f(u, a_0, a_1, a_2)$
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		
a_3	$f(a_3)$			

Formula (5) may evidently be generalised to express a function whose divided differences of order $(n+1)$ are negligible or zero, in the form

$$\begin{aligned} f(u) = f(a_0) &+ (u-a_0)f(a_0, a_1) + (u-a_0)(u-a_1)f(a_0, a_1, a_2) \\ &+ (u-a_0)(u-a_1)(u-a_2)f(a_0, a_1, a_2, a_3) + \ldots \\ &+ (u-a_0)(u-a_1)\ldots(u-a_{n-1})f(a_0, a_1, \ldots, a_n). \qquad (6) \end{aligned}$$

This formula was discovered by Newton.†

The first term on the right-hand side of this equation represents the polynomial of zero degree, which has the value $f(a_0)$ at $u = a_0$. The first two terms together represent the polynomial of degree 1, which has the values $f(a_0)$ and $f(a_1)$ at a_0 and a_1 respectively, and so on.

The remainder term which must be added to the right-hand side of the equation in order to obtain strict accuracy is in fact

$$(u-a_0)(u-a_1)\ldots(u-a_n)f(u, a_0, a_1, \ldots, a_n).$$

But this term vanishes if the divided differences of order n are rigorously constant.

Ex.—From the table given below to find the entry corresponding to 3·7608.

x.	$f(x)$.		
$a_0 = 0$	·3989423		
		− 500	
$a_1 = 2·5069$	·3988169		− 199
		− 1499	
$a_2 = 5·0154$	·3984408		− 199
		− 2496	
$a_3 = 7·5270$	·3978138		

---

* In practice the value of $f(u)$ is usually found by forming the successive divided differences in this way, as in the worked-out example below.

† *Principia* (1687), Book iii. Lemma v. Case ii. Cf. Cauchy, *Œuvres*, (1) 5, p. 409.

Forming the successive divided differences of $f(u)$, where $u = 3{\cdot}7608$, we find

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) = -199,$$
$$f(u, a_0) = -500 + 1{\cdot}2539 \times (-199) = -749{\cdot}526,$$
$$f(u) = {\cdot}3989423 + 3{\cdot}7608 \times (-749{\cdot}526).$$

The calculated value is therefore 0·3986604.

14. The Gregory-Newton Formula as a Special Case of Newton's Formula.—The Gregory-Newton formula may be regarded as the special case of the formula of the last section when the intervals of the argument are equal.

For in Newton's formula for unequal intervals suppose that we put

$$a_0 = a, \quad a_1 = a + w, \quad a_2 = a + 2w, \; . \; . \; ., \quad u = a + xw.$$

By constructing a table of divided differences, we see that

$$f(a_0, a_1) = \frac{1}{w}\Delta f(a), \quad f(a_1, a_2) = \frac{1}{w}\Delta f(a + w),$$

$$\therefore \; f(a_0, a_1, a_2) = \frac{1}{2\,!\,w^2}\Delta^2 f(a).$$

In the same way we find

$$f(a_0, a_1, a_2, a_3) = \frac{1}{3\,!\,w^3}\Delta^3 f(a),$$

and so on.

If we now replace u by $a + xw$, the formula for unequal intervals of the argument becomes

$$f(a + xw) = f(a) + x\Delta f(a) + \frac{x(x-1)}{2\,!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(a) + . \; . \; .$$

which is the Gregory-Newton formula.

15. The Practical Application of Newton's Formula.—In laboratory computation from Newton's formula, we proceed by a method which is really identical with that given above (Ex. § 13). Rearranging the formula of § 13, we see that

$$f(u) = f(a_0) + (u - a_0)\,[f(a_0, a_1)$$
$$+ (u - a_1)\{f(a_0, a_1, a_2) + (u - a_2)(f(a_0, a_1, a_2, a_3) + . \; . \; .)\}].$$

This equation may be written in the form

$$f(u) = f(a_0) + (u - a_0)v_1, \qquad (1)$$

$$\text{where} \begin{cases} v_1 = f(a_0, a_1) + (u - a_1)v_2, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \\ v_r = r\text{th divided difference} + (u - a_r)v_{r+1}, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \\ v_n = f(a_0, a_1, \; . \; . \; ., a_n), \text{ a constant.} \end{cases}$$

The v's are computed in the following order: v_{n-1}, v_{n-2}, . . . v_1. The value of $f(u)$ is then obtained from equation (1).

Ex.—To find the function corresponding to the argument 6·417 in the following difference table:

Argument.	*Entry.*			
$a_0 = 5$	150			
		121		
$a_1 = 7$	392		24	
		265		1
$a_2 = 11$	1452		32	
		457		1
$a_3 = 13$	2366		46	
		917		
$a_4 = 21$	9702			

$$u = 6{\cdot}417, \quad v_3 = 1, \quad v_2 = 24 + (6{\cdot}417 - 11)1 = 19{\cdot}417,$$
$$v_1 = 121 + (6{\cdot}417 - 7)19{\cdot}417 = 109{\cdot}679889,$$
$$\therefore\ f(6{\cdot}417) = 150 + (6{\cdot}417 - 5)109{\cdot}679889$$
$$= 305{\cdot}416402713.$$

16. **Divided Differences with Repeated Arguments.**—The original definition of divided differences presupposes that the arguments concerned are all different. If, however, the quantity $f(a_0, a_0 + \epsilon)$ tends to a definite limit as ϵ tends to zero, we denote this limit by $f(a_0, a_0)$, and similarly for divided differences of higher order.

Now suppose that in § 13, $u = a_0$. Since the differences of order 3 are supposed constant, we see that $f(a_0, a_0, a_1, a_2)$ is equal to $f(a_0, a_1, a_2, a_3)$, and the remaining differences $f(a_0, a_0, a_1)$, $f(a_0, a_0)$ may then be calculated just as in the general case when u and a_0 were supposed different. We may now form another set of differences by again taking $u = a_0$. Repeating this method, we obtain the following table of divided differences:

Argument.	*Entry.*			
a_0	$f(a_0)$			
		$f(a_0, a_0)$		$f(a_0, a_0, a_0, a_0)$
a_0	$f(a_0)$		$f(a_0, a_0, a_0)$	
		$f(a_0, a_0)$		$f(a_0, a_0, a_0, a_1)$
a_0	$f(a_0)$		$f(a_0, a_0, a_1)$	
		$f(a_0, a_1)$		$f(a_0, a_0, a_1, a_2)$
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		
a_3	$f(a_3)$			

In terms of these divided differences with repeated arguments the formula of Newton becomes

$$f(u) = f(a_0) + (u - a_0)f(a_0, a_0) + (u - a_0)^2 f(a_0, a_0, a_0) + (u - a_0)^3 f(a_0, a_0, a_0, a_0) + \dots$$

This formula will be used later to obtain an expression for the derivatives of a function in terms of its divided differences.[*]

Ex.—Given the values

x	5	11	27	34	42
$f(x)$	23	899	17315	35606	68510

to find $f(x)$ in terms of powers of $(x - 3)$.

Constructing a table of divided differences and extending it to include repeated arguments for $x = 3$, we obtain

x.	$f(x)$.			
42	68510			
		4113		
34	35606		100	
		2613		1
27	17315		69	
		1026		1
11	899		40	
		146		1
5	23		16	
		18		1
3	− 13		8	
		2		1
3	− 13		6	
		2		
3	− 13			

Applying Newton's formula for repeated arguments, the required value is $f(x) = -13 + 2(x - 3) + 6(x - 3)^2 + (x - 3)^3$.

17. Lagrange's Formula of Interpolation.—Let $f(x)$ be the polynomial of degree n which for values $a_0, a_1, a_2, \dots, a_n$ of the argument x has the values $f(a_0), f(a_1), \dots, f(a_n)$ respectively. By the definition of divided differences, we have

$$\begin{aligned} f(a_0, a_1, a_2, \dots, a_n, x) &= \frac{f(x)}{(x - a_0)(x - a_1)\dots(x - a_n)} + \frac{f(a_0)}{(a_0 - x)(a_0 - a_1)\dots(a_0 - a_n)} \\ &+ \frac{f(a_1)}{(a_1 - x)(a_1 - a_0)\dots(a_1 - a_n)} + \dots \\ &+ \frac{f(a_n)}{(a_n - x)(a_n - a_0)\dots(a_n - a_{n-1})}. \end{aligned}$$

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[*] § 37.

Since $f(x)$ is a polynomial of degree n, its divided differences of order $(n+1)$ are zero, *i.e.*

$$f(a_0, a_1\, a_2, \ldots a_n, x) = 0.$$

Arranging the factors of the denominators in the above fractions so that the first factor in each denominator is of the form $(x - a_p)$, we obtain

$$\begin{aligned}\frac{f(x)}{(x-a_0)(x-a_1)\ldots(x-a_n)} &= \frac{f(a_0)}{(x-a_0)(a_0-a_1)(a_0-a_2)\ldots(a_0-a_n)}\\ &+\frac{f(a_1)}{(x-a_1)(a_1-a_0)\ldots(a_1-a_n)}\\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &+\frac{f(a_n)}{(x-a_n)(a_n-a_0)\ldots(a_n-a_{n-1})},\end{aligned}\qquad\text{(A)}$$

which is Lagrange's formula in a form suitable for computation.*

Another way of writing this formula is obtained by multiplying both sides of equation (A) by

$$(x-a_0)(x-a_1)(x-a_2)\ldots(x-a_n),$$

when we obtain

$$\begin{aligned}f(x) &= \frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(a_0-a_1)(a_0-a_2)\ldots(a_0-a_n)}f(a_0)\\ &+\frac{(x-a_0)(x-a_2)\ldots(x-a_n)}{(a_1-a_0)(a_1-a_2)\ldots(a_1-a_n)}f(a_1)\\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &+\frac{(x-a_0)(x-a_1)\ldots(x-a_{n-1})}{(a_n-a_0)(a_n-a_1)\ldots(a_n-a_{n-1})}f(a_n).\end{aligned}\qquad\text{(B)}$$

It is important to note that when a set of experimental data obey a law which can be expressed algebraically as a polynomial of degree n, then not less than $(n+1)$ observations are required in order to construct the polynomial. If only n values were used, the resulting polynomial would be of degree $(n-1)$. Before applying the Lagrange formula it is therefore necessary to ascertain the order of the divided differences which are of constant value and thus find the proper value for n.

Ex. 1.—*Given the values* $\begin{array}{c|cc|cc} x & 14 & 17 & 31 & 35\\ f(x) & 68{\cdot}7 & 64{\cdot}0 & 44{\cdot}0 & 39{\cdot}1\end{array}$ *to calculate the value of* $f(x)$ *corresponding to* $x = 27$.

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* This formula, generally known as Lagrange's, was discovered by E. Waring, *Phil. Trans.* **69** (1779), p. 59.

Applying formula (A), we obtain

$$\frac{f(27)}{(27-14)(27-17)(27-31)(27-35)}$$

$$=\frac{68\cdot7}{(27-14)(14-17)(14-31)(14-35)}$$

$$+\frac{64\cdot0}{(27-17)(17-14)(17-31)(17-35)}$$

$$+\frac{44\cdot0}{(27-31)(31-14)(31-17)(31-35)}$$

$$+\frac{39\cdot1}{(27-35)(35-14)(35-17)(35-31)}$$

or

$$\frac{f(27)}{4160}=-\frac{68\cdot7}{13923}+\frac{64\cdot0}{7560}+\frac{44\cdot0}{3808}-\frac{39\cdot1}{12096},$$

$$\therefore f(27)=49\cdot317 \text{ (approx.)}.$$

The required value is 49·3.

Ex. 2.—*Given the data* $\begin{matrix} x & 0 & 1 & 2 & 5 \\ f(x) & 2 & 3 & 12 & 147 \end{matrix}$, *to form the cubic function of* x.

Applying formula (B), we have

$$f(x)=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}2+\frac{x(x-2)(x-5)}{1(1-2)(1-5)}3+\frac{x(x-1)(x-5)}{2(2-1)(2-5)}12$$
$$+\frac{x(x-1)(x-2)}{5(5-1)(5-2)}147$$
$$=x^3+x^2-x+2.$$

18. **An alternative proof of Lagrange's formula by the use of determinants** is the following:

Let P_n denote a polynomial of degree n, and put

$$P_n=A+Bx+Cx^2+\ldots+Lx^n$$
$$=f(x).$$

Substituting in succession the values $a_0, a_1, \ldots, a_n$ for x, we obtain

$$f(a_0)=A+Ba_0+Ca_0^2+\ldots+La_0^n,$$
$$f(a_1)=A+Ba_1+Ca_1^2+\ldots+La_1^n,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f(a_n)=A+Ba_n+Ca_n^2+\ldots+La_n^n.$$

Eliminating A, B, C, . . . from these equations determinantally we have

$$0=\begin{vmatrix} P_n & f(a_0) & f(a_1) & f(a_2) & \dots & f(a_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ x & a_0 & a_1 & a_2 & \dots & a_n \\ x^2 & a_0^2 & a_1^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & a_1^n & a_2^n & \dots & a_n^n \end{vmatrix}$$

Expanding this determinant according to the elements of the first row, we see that

$$P_n\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ a_0^2 & a_1^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = f(a_0)\begin{vmatrix} 1 & 1 & \dots & 1 \\ x & a_1 & \dots & a_n \\ x^2 & a_1^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ x^n & a_1^n & \dots & a_n^n \end{vmatrix}$$

$$-f(a_1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x & a_0 & a_2 & \dots & a_n \\ x^2 & a_0^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & a_2^n & \dots & a_n^n \end{vmatrix} + \dots + (-1)^n f(a_n)\begin{vmatrix} 1 & 1 & \dots & 1 \\ x & a_0 & \dots & a_{n-1} \\ x^2 & a_0^2 & \dots & a^2_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & \dots & a^n_{n-1} \end{vmatrix} \qquad (1)$$

The determinants in this equation may be represented as difference-products. The coefficient of $f(a_0)$ is the difference-product of $x, a_1, \dots, a_n$, the coefficient of $f(a_1)$ is the difference-product of $x, a_0, a_2, \dots, a_n$, and so on. We may write

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_0 & a_2 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^n & a_0^n & a_2^n & \dots & a_n^n \end{vmatrix} = +\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_2 & a_0 & a_1 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2^n & a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = - \dots$$

i.e. the coefficient of P_n is equal to the difference-product of $a_0, a_1, \dots, a_n$: it is also equal to minus the difference-product of $a_1, a_0, a_2, \dots, a_n$, or to plus the difference-product of $a_2, a_0, a_1, \dots, a_n$, and so on. If we now divide throughout by the coefficient of P_n in equation (1), we obtain the result:

$$\begin{aligned}
P_n = {} & f(a_0)\frac{(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)}{(a_0-a_1)(a_0-a_2)(a_0-a_3)\ldots(a_0-a_n)} \\
& + f(a_1)\frac{(x-a_0)(x-a_2)(x-a_3)\ldots(x-a_n)}{(a_1-a_0)(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)} \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& + f(a_n)\frac{(x-a_0)(x-a_1)\ldots(x-a_{n-1})}{(a_n-a_0)(a_n-a_1)\ldots(a_n-a_{n-1})},
\end{aligned}$$

which is the formula required.

There is an infinite number of functions of x, each of which has the values $f(a_0)$, $f(a_1)$, . . ., $f(a_n)$ at a_0, a_1, . . ., a_n respectively. In the practical applications of mathematics, however, we consider only functions, such that if a_0, a_1, . . ., a_n are sufficiently close together, any one of the functions may be represented with tolerable accuracy by the polynomial P_n, for the range of values included between a_0, a_1, . . ., a_n. The formula may thus be used for interpolation.

19. The Remainder Term in Lagrange's Formula of Interpolation.*—Let $f(x)$ be a function of the real variable x defined in an interval to which belong the values x_0, x_1, . . ., x_n, and possessing in this interval the derivative of order n.

Consider the function $g(x)$, where

$$g(x) = \begin{vmatrix} f(x) & x^n & x^{n-1} & \ldots & x & 1 \\ f(x_0) & x_0^n & x_0^{n-1} & \ldots & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_n) & x_n^n & x_n^{n-1} & \ldots & x_n & 1 \end{vmatrix}$$

The determinant vanishes for the values x_0, x_1, . . ., x_n. By the differential calculus we see that since $g(x)$ vanishes for $(n+1)$ values of x, its derivative $g'(x)$ vanishes for n values of x, the second derivative for $(n-1)$ values, and so on; the nth derivative vanishing for one value of x in the interval. Thus there exists a value x intermediate between x_0, x_1, . . ., x_n such that $g^{(n)}(x) = 0$.

Forming the nth derivative of the determinant by differentiating the variable elements of the first row, we have:

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* Peano, *Scritti offerti ad E. d' Ovidio* (Turin, 1918), p. 333.

$$\begin{vmatrix} f^n(x) & n! & 0 & \dots & 0 & 0 \\ f(x_0) & x_0^n & x_0^{n-1} & \dots & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_n) & x_n^n & x_n^{n-1} & \dots & x_n & 1 \end{vmatrix} = 0.$$

If we expand this determinant according to the elements of the first column and solve for $f(x_0)$ in the resulting equation, we find

$$\begin{aligned} f(x_0) = {} & \frac{(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n)}{(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n)} f(x_1) \\ & + \frac{(x_0 - x_1)(x_0 - x_3) \dots (x_0 - x_n)}{(x_2 - x_1)(x_2 - x_3) \dots (x_2 - x_n)} f(x_2) \\ & \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ & + \frac{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_{n-1})}{(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1})} f(x_n) \\ & + \frac{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)}{n!} f^{(n)}(x), \end{aligned}$$

where x is some number intermediate between $x_0, x_1, \dots, x_n$. This is *Lagrange's formula with a remainder term.*

EXAMPLES ON CHAPTER II

1. If $f(x) = \frac{1}{x^2}$, find the divided differences $f(a, b), f(a, b, c)$, and $f(a, b, c, d)$.

2. If $f(x) = g(x) + h(x)$, where $g(x) = x^4$ and $h(x) = x^3$, verify that $f(5, 7, 11, 13) = g(5, 7, 11, 13) + h(5, 7, 11, 13)$.

3. Given the values

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

form the table of divided differences and extend it to include the values of the function for $x = 3$ and $x = 14$.

4. Find the function $f(x)$ in each of the following cases:

(*a*)

x	11	13	14	18	19	21
$f(x)$	1342	2210	2758	5850	6878	9282

(*b*)

x	16	17	19	23	29	31
$f(x)$	65536	83521	130321	279841	707281	923521

by means of a table of divided differences.

5. Calculate $f(1)$, given the values

x	0	2	3	6	7	9
$f(x)$	658503	704969	729000	804357	830584	884736

6. Assuming $f(x)$ to be a function of the fourth degree in x, find the value of $f(19)$ from the values

x	11	17	21	23	31
$f(x)$	14646	83526	194486	279846	923526

7. The values of a cubic function are 150, 392, 1452, 2366, and 5202, corresponding to the values of the argument 5, 7, 11, 13, 17 respectively. Apply the Lagrange formula to find the function when the argument has the values 9 and 6·5 respectively.

8. Find an expression for the function in each of the examples (6) and (7), using the Lagrange formula of interpolation.

9. If
$$\varphi(x) = f(x)g(x),$$
obtain the formula (analogous to Leibnitz's formula for the nth derivative of a product)
$$\varphi(x_n, x_{n-1}, \ldots, x_0) = \sum_{k=0}^{n} f(x_n, \ldots, x_k)g(x_k, \ldots x_0).$$
(Steffensen.)

CHAPTER III

CENTRAL-DIFFERENCE FORMULAE

20. Central-Difference Notations.—In this chapter we shall consider certain formulae of interpolation which employ differences taken nearly or exactly from a single horizontal line of the difference table. In order to express these simply it is convenient to modify the notation of the calculus of differences.

Several systems of modified notation are in use. One, which we shall frequently employ, was introduced by W. F. Sheppard* and will be understood from the following difference table. It is based on a symbol δ which may be regarded as equivalent to $\Delta E^{-\frac{1}{2}}$, where E as usual denotes the transition from any number to the number next below it in the difference table, *i.e.* $E = 1 + \Delta$.

Since $\delta \equiv \Delta E^{-\frac{1}{2}}$ and therefore $\Delta \equiv \delta E^{\frac{1}{2}}$, we may write $\Delta u_0 = \delta u_{\frac{1}{2}}$, $\Delta^2 u_0 = \delta^2 u_1$, $\Delta^3 u_0 = \delta^3 u_{\frac{3}{2}}, \ldots, \Delta^n u_0 = \delta^n u_{\frac{n}{2}}$, and so on. Rewriting the ordinary difference table, we obtain

Argument.	*Entry.*				
$a - 2w$	u_{-2}				
		$\delta u_{-\frac{3}{2}}$			
$a - w$	u_{-1}		$\delta^2 u_{-1}$		
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
a	u_0		$\delta^2 u_0$		$\delta^4 u_0$
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
$a + w$	u_1		$\delta^2 u_1$		$\delta^4 u_1$
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$	
$a + 2w$	u_2		$\delta^2 u_2$		$\delta^4 u_2$

If we suppose each row of the difference table to be numbered with the suffix p of the corresponding entry u_p, or, in the case of a row situated midway between two entries u_p and u_{p+1}, to take the number $p + \frac{1}{2}$, we see that $\Delta^{2r} u_0$, the differences of even order of u_0, are represented in the central-difference notation by $\delta^{2r} u_r$, since they are situated in the row r. The differences of odd order $\Delta^{2r+1}u_0$ are represented by the expression $\delta^{2r+1}u_{r+\frac{1}{2}}$ since they lie in the row $r+\frac{1}{2}$.

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* *Proc. London Math. Soc.* **31** (1899), p. 459.

It is often required to find the arithmetic mean of two adjacent entries in the same column of differences. In the δ system of notation we indicate this mean by the symbol μ. Thus $\mu\delta u_0$ is defined to be $\frac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}})$, $\mu\delta^3 u_0$ is $\frac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}})$, and so on for the mean differences of the other entries. The mean differences may be inserted in the table to fill in the gaps that occur between the symbols of the quantities from which they are derived.

In another notation which was suggested by S. A. Joffe* the symbol $\triangle\!\!\!\!\!{\scriptscriptstyle c}\;$ is used instead of δ. The notation is illustrated in the following difference table:

Argument.	*Entry.*				
$a-2w$	u_{-2}				
		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\; u_{-\frac{3}{2}}$			
$a-w$	u_{-1}		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^2 u_{-1}$		
		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\; u_{-\frac{1}{2}}$		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^3 u_{-\frac{1}{2}}$	
a	u_0		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^2 u_0$		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^4 u_0$
		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\; u_{\frac{1}{2}}$		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^3 u_{\frac{1}{2}}$	
$a+w$	u_1		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\;^2 u_1$		
		$\triangle\!\!\!\!\!{\scriptscriptstyle c}\; u_{\frac{3}{2}}$			
$a+2w$	u_2				

21. The Newton-Gauss Formula of Interpolation.—Suppose that a function $f(a+xw)$ is given for the values

$$\ldots\; a-w,\; a,\; a+w,\; a+2w,\; \ldots$$

of its argument.

If in the Newton formula for unequal intervals we take $a_0 = a$, $a_1 = a+w$, $a_2 = a-w$, $a_3 = a+2w$, $a_4 = a-2w$, and so on, and denote $a+xw$ by u, we obtain

$$\begin{aligned} f(u) = f(a) &+ (u-a)f(a, a+w) + (u-a)(u-a-w)f(a, a+w, a-w) \\ &+ (u-a)(u-a-w)(u-a+w)f(a, a+w, a-w, a+2w) \\ &+ (u-a)(u-a-w)(u-a+w)(u-a-2w) \\ &\qquad\qquad f(a, a+w, a-w, a+2w, a-2w) \\ &+ (u-a)(u-a-w)(u-a+w)(u-a-2w)(u-a+2w) \\ &\qquad\qquad f(a, a+w, a-w, a+2w, a-2w, a+3w). \\ &+ \ldots \end{aligned} \tag{1}$$

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* *Trans. Act. Soc. Amer.* 18 (1917), p. 91.

The divided differences contained in this equation may be written in the ordinary notation of differences as follows:

$$f(a,\ a+w)=\frac{1}{w}\Delta f(a),$$

$$f(a,\ a+w,\ a-w)=\frac{1}{2!\,w^2}\Delta^2 f(a-w),$$

$$f(a,\ a+w,\ a-w,\ a+2w)=\frac{1}{3!\,w^3}\Delta^3 f(a-w),$$

etc.

Equation (1) thus takes the form

$$\begin{aligned}f(a+xw)=f(a)+x\Delta f(a)&+\frac{x(x-1)}{2!}\Delta^2 f(a-w)\\&+\frac{(x+1)x(x-1)}{3!}\Delta^3 f(a-w)\\&+\frac{(x+1)x(x-1)(x-2)}{4!}\Delta^4 f(a-2w)\\&+\frac{(x+2)(x+1)x(x-1)(x-2)}{5!}\Delta^5 f(a-2w)+\ .\ .\ .\qquad\text{(A)}\end{aligned}$$

This formula, which is one of the group of formulae known to Newton, is often called the *Gauss* formula.

The differences used in this formula are as nearly as possible in the horizontal line through $f(a)$ in the original difference table. The formula is therefore convenient for use when the value of the argument for which the function is required is near the middle of the tabulated values. This formula may be represented more simply by using the symbol $(n)_r$ to denote the binomial coefficient

$$\frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+1)}{r!},$$

so that it may be written

$$\begin{aligned}f(a+xw)=f(a)+x\Delta f(a)+(x)_2\Delta^2 f(a-w)+(x+1)_3\Delta^3 f(a-w)&\\+(x+1)_4\Delta^4 f(a-2w)+(x+2)_5\Delta^5 f(a-2w)+\ .\ .\ .\ .&\qquad\text{(B)}\end{aligned}$$

22. **The Newton-Gauss Backward Formula.**—From the formula of the last section another may be derived which is often used when x is measured in a negative direction from $f(a)$, *i.e.* towards decreasing values of the argument. Suppose we write $f(a-xw)$ in the form $f\{a+x(-w)\}$ and change the sign of w in the discussion of the last section. The

order of the arguments and corresponding entries is then reversed. Instead of $\Delta f(a)$ in the Newton-Gauss formula we now have $f(a-w)-f(a)$, or $-\Delta f(a-w)$; $\Delta^3 f(a-w)$ in the above formula becomes $-\Delta^3 f(a-2w)$; $\Delta^5 f(a-2w)$ becomes $-\Delta^5 f(a-3w)$, and so on. We thus obtain the formula

$$\begin{aligned} f(a-xw) = f(a) - x\Delta f(a-w) + (x)_2\Delta^2 f(a-w) - (x+1)_3\Delta^3 f(a-2w) \\ + (x+1)_4\Delta^4 f(a-2w) - (x+2)_5\Delta^5 f(a-3w) + \ldots \end{aligned}$$

which has been called the *Newton-Gauss formula for negative interpolation*, or the *Newton-Gauss backward formula.*

23. **The Newton-Stirling Formula.**—In the Gauss formula

$$\begin{aligned} f(a+xw) &= f(a) + x\Delta f(a) + \tfrac{1}{2}x(x-1)\Delta^2 f(a-w) \\ &\quad + \tfrac{1}{6}(x+1)x(x-1)\Delta^3 f(a-w) \\ &\quad + \tfrac{1}{24}(x+1)x(x-1)(x-2)\Delta^4 f(a-2w) + \ldots \end{aligned}$$

the terms may be rearranged thus:

$$\begin{aligned} f(a+xw) &= f(a) + x\{\Delta f(a) - \tfrac{1}{2}\Delta^2 f(a-w)\} + \frac{x^2}{2!}\Delta^2 f(a-w) \\ &\quad + \frac{x(x^2-1^2)}{3!}\{\Delta^3 f(a-w) - \tfrac{1}{2}\Delta^4 f(a-2w)\} \\ &\quad + \frac{x^2(x^2-1^2)}{4!}\Delta^4 f(a-2w) + \ldots \end{aligned}$$

Suppose we replace the differences of even order within the brackets by differences of odd order, using the identities

$$\begin{aligned} \Delta^2 f(a-w) &= \Delta f(a) - \Delta f(a-w), \\ \Delta^4 f(a-2w) &= \Delta^3 f(a-w) - \Delta^3 f(a-2w), \end{aligned}$$

and so on. We obtain the result

$$\begin{aligned} f(a+xw) &= f(a) + x\frac{\Delta f(a) + \Delta f(a-w)}{2} + \frac{x^2}{2!}\Delta^2 f(a-w) \\ &\quad + \frac{x(x^2-1^2)}{3!}\,\frac{\Delta^3 f(a-w) + \Delta^3 f(a-2w)}{2} + \frac{x^2}{4!}(x^2-1^2)\Delta^4 f(a-2w) \\ &\quad + \frac{x(x^2-1^2)(x^2-2^2)}{5!}\,\frac{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)}{2} \\ &\quad + \frac{x^2(x^2-1^2)(x^2-2^2)}{6!}\Delta^6 f(a-3w) + \ldots \qquad \text{(A)} \end{aligned}$$

This formula, which was first given by Newton,* was afterwards studied by Stirling † and is called the *Newton-Stirling formula.*

The mean-differences $\frac{1}{2}\{\Delta f(a)+\Delta f(a-w)\}$, $\frac{1}{2}\{\Delta^3 f(a-2w)+\Delta^3 f(a-w)\}$, etc., are completely symmetrical with regard to increasing and decreasing arguments. This fact enables us to express the formula very concisely by means of the central-difference notation of § 20 :

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1)}{4!}\delta^4 u_0 + \frac{x(x^2-1)(x^2-2^2)}{5!}\mu\delta^5 u_0 + \dots \quad \text{(B)}$$

where $\mu\delta u_0 = \frac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}})$,
$\mu\delta^3 u_0 = \frac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}})$,
and so on.

24. The Newton-Bessel Formula.—In the Newton-Gauss formula

$$\begin{aligned} f(a+xw) &= f(a) + x\Delta f(a) + \tfrac{1}{2}x(x-1)\Delta^2 f(a-w) \\ &\quad + \tfrac{1}{6}(x+1)x(x-1)\Delta^3 f(a-w) \\ &\quad + \tfrac{1}{24}(x+1)x(x-1)(x-2)\Delta^4 f(a-2w) + \dots; \end{aligned}$$

let us substitute for $\frac{1}{2}f(a)$, $\frac{1}{2}\Delta^2 f(a-w)$, $\frac{1}{2}\Delta^4 f(a-2w)$, etc., their values obtained from the identities

$$\begin{aligned} f(a) &= f(a+w) - \Delta f(a), \\ \Delta^2 f(a-w) &= \Delta^2 f(a) - \Delta^3 f(a-w), \\ \Delta^4 f(a-2w) &= \Delta^4 f(a-w) - \Delta^5 f(a-2w), \\ &\text{etc.} \end{aligned}$$

The above equation becomes

$$\begin{aligned} f(a+xw) &= \tfrac{1}{2}\{f(a)+f(a+w)\} + (x-\tfrac{1}{2})\Delta f(a) \\ &\quad + \frac{x(x-1)}{2!}\tfrac{1}{2}\{\Delta^2 f(a-w) + \Delta^2 f(a)\} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\Delta^3 f(a-w) \\ &\quad + \frac{(x+1)x(x-1)(x-2)}{4!}\cdot\tfrac{1}{2}\{\Delta^4 f(a-2w) + \Delta^4 f(a-w)\} + \dots \quad \text{(A)} \end{aligned}$$

which is symmetrical with respect to the argument $(a+\frac{1}{2}w)$.

This formula, which was first given by Newton ‡ and later used by Bessel, is called the *Newton-Bessel formula.*

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* Newton, *Methodus Differentialis* (1711), Prop. iii. Case i.

† Stirling, *Methodus Differentialis* (1730), Prop. xx.

‡ *Methodus Differentialis* (1711), Prop. iii. Case ii. ; Stirling, *Methodus Differentialis* (1730), Prop. xx. Case ii.

If in this formula we write $x - \frac{1}{2} = y$, it becomes

$$\begin{aligned} f(a + \tfrac{1}{2}w + yw) &= \tfrac{1}{2}\{f(a) + f(a + w)\} + y\Delta f(a) \\ &\quad + \frac{y^2 - \frac{1}{4}}{2!}\tfrac{1}{2}\{\Delta^2 f(a - w) + \Delta^2 f(a)\} + \frac{y(y^2 - \frac{1}{4})}{3!}\Delta^3 f(a - w) \\ &\quad + \frac{(y^2 - \frac{1}{4})(y^2 - \frac{9}{4})}{4!}\tfrac{1}{2}\{\Delta^4 f(a - 2w) + \Delta^4 f(a - w)\} + \ldots \qquad \text{(B)} \end{aligned}$$

25. The Laplace-Everett Formula.—When it is required to interpolate between $f(a)$ and $f(a + w)$ in the construction of tables by the subdivision of intervals, statisticians often use a formula due to Laplace and Everett,* which may be written

$$\begin{aligned} u_x &= \left[\xi + \frac{\xi(\xi^2 - 1)}{3!}\delta^2 + \frac{\xi(\xi^2 - 1)(\xi^2 - 4)}{5!}\delta^4 + \ldots\right]u_0 \\ &\quad + \left[x + \frac{x(x^2 - 1)}{3!}\delta^2 + \frac{x(x^2 - 1)(x^2 - 4)}{5!}\delta^4 + \ldots\right]u_1, \end{aligned}$$

where u_x denotes $f(a + xw)$, and ξ denotes $(1 - x)$, and where as usual δ^2 denotes $\Delta^2 E^{-1}$. Thus for $u_{\frac{3}{4}}$, $x = \frac{3}{4}$, $\xi = 1 - \frac{3}{4} = \frac{1}{4}$.

This formula involves only even central differences of each of the two middle terms of the series between which the interpolation has to be made.

To prove this formula we eliminate from the Newton-Gauss formula

$$\begin{aligned} f(a + xw) &= f(a) + x\Delta f(a) + (x)_2\Delta^2 f(a - w) + (x + 1)_3\Delta^3 f(a - w) \\ &\quad + (x + 1)_4\Delta^4 f(a - 2w) + (x + 2)_5\Delta^5 f(a - 2w) + \ldots \end{aligned}$$

the differences of odd order by means of the relations

$$\begin{aligned} \Delta f(a) = f(a + w) - f(a), \quad \Delta^3 f(a - w) &= \Delta^2 f(a) - \Delta^2 f(a - w), \\ \Delta^5 f(a - 2w) &= \Delta^4 f(a - w) - \Delta^4 f(a - 2w) \ \ldots \end{aligned}$$

The Newton-Gauss formula becomes

$$\begin{aligned} f(a + xw) &= f(a) + x\{f(a + w) - f(a)\} + (x)_2\Delta^2 f(a - w) \\ &\quad + (x + 1)_3\{\Delta^2 f(a) - \Delta^2 f(a - w)\} + (x + 1)_4\Delta^4 f(a - 2w) \\ &\quad + (x + 2)_5\{\Delta^4 f(a - w) - \Delta^4 f(a - 2w)\} + \ldots \end{aligned}$$

Using the relation $(p + 1)_{q+1} = (p)_{q+1} + (p)_q$, this equation may be written

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* Laplace, *Théorie anal. des Prob.*, p. 15; Everett, *Brit. Assoc. Rep.* (1900), p. 648; *J.I.A.* 35, p. 452 (1901). Tables of the co-efficients have been published in *Tracts for Computers*, No. V.

$$f(a+xw) = (1-x)f(a) + xf(a+w) + (x+1)_3\Delta^2 f(a) - (x)_3\Delta^2 f(a-w) \\ + (x+2)_5\Delta^4 f(a-w) - (x+1)_5\Delta^4 f(a-2w) + \dots$$

Introducing central differences and rearranging the terms,

$$f(a+xw) = (1-x)f(a) - (x)_3\delta^2 f(a) - (x+1)_5\delta^4 f(a) - \dots \\ + xf(a+w) + (x+1)_3\delta^2 f(a+w) + (x+2)_5\delta^4 f(a+w) + \dots$$

If we now transform the coefficients of $f(a)$ by means of the relation $1-x=\xi$, so that $(x)_3 = -(\xi+1)_3$, $(x+1)_5 = -(\xi+2)_5$, etc., we have

$$f(a+xw) = \xi f(a) + (\xi+1)_3\delta^2 f(a) + (\xi+2)_5\delta^4 f(a) + \dots \\ + xf(a+w) + (x+1)_3\delta^2 f(a+w) + (x+2)_5\delta^4 f(a+w) + \dots$$

which is *Everett's formula for equal* intervals of the argument.*

26. Example of Central-Difference Formulae.—The following example illustrates the various central-difference formulae:†

To compute the value of $\log_{10}$ *cosh* 0·3655, *having given a table of values of* $\log_{10}$ *cosh* x *at intervals* 0·002 *of the argument.*

Forming the difference table, we see that the differences of the third order are approximately constant. The differences of the fourth order will, however, be taken into account since such a difference may affect the accuracy of the last figure of the result.

Argument.	*Entry.*				
0·360	0·0275 5462 3980				
		30061 3825			
0·362	278 5523 7805		152 8035		
		30214 1860		−2122	
0·364	281 5737 9665		152 5913		−13
		30366 7773		−2135	
0·366	284 6104 7438		152 3778		−3
		30519 1551		−2138	
0·368	287 6623 8989		152 1640		
		30671 3191			
0·370	290 7295 2180				

In *Everett's* formula put $x=\frac{3}{4}$, $\xi=\frac{1}{4}$, and $u_0 = 0·0281\ 5737\ 9665$.

$$f(0·3655) = \tfrac{1}{4}(281\ 5737\ 9665) + (-\tfrac{5}{128})(152\ 5913) + \tfrac{63}{8192}(-13) \\ + \tfrac{3}{4}(284\ 6104\ 7438) + (-\tfrac{7}{128})(152\ 3778) + \tfrac{77}{8192}(-3) \\ = 283\ 8513\ 0494·75 - 14\ 2937·59 - 0·13 = 283\ 8498\ 7557·03.$$

$\therefore$ log cosh (0·3655) = $\underline{0·0283\ 8498\ 7557}$.

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* Corresponding formulae for unequal intervals have been given by R. Todhunter, *J.I.A.* **50** (1916), p. 137, and by G. J. Lidstone, *Proc. Edin. Math. Soc.* **40** (1922), p. 26.

† A valuable set of worked-out examples is given by L. J. Comrie in the *Nautical Almanac* for 1937, pp. 931–934.

In the *Newton-Bessel* formula put $x=\frac{3}{4}$.

$$\begin{aligned} f(0\cdot3655) &= \tfrac{1}{2}\begin{pmatrix} 281\ 5737\ 9665 \\ +\ 284\ 6104\ 7438 \end{pmatrix} + \tfrac{1}{4}(30366\ 7773) \\ &\quad + (-\tfrac{3}{32})\tfrac{1}{2}\begin{pmatrix} 152\ 5913 \\ +\ 152\ 3778 \end{pmatrix} - \tfrac{1}{128}(-2135) + \tfrac{35}{2048}\tfrac{1}{2}(-13-3) \\ &= 2830921\,3551\cdot5 + 7591\,6943\cdot25 - 142954\cdot27 + 16\cdot68 - 0\cdot14 \\ &= 283\ 8498\ 7557\cdot02. \end{aligned}$$

$\therefore$ log cosh $(0\cdot3655) = \underline{0\cdot0283\ 8498\ 7557}$.

By the *Newton-Gauss* formula

$$\begin{aligned} f(0\cdot3655) &= 281\ 5737\ 9665 + \tfrac{3}{4}(3\ 0366\ 7773) + (-\tfrac{3}{32})(152\ 5913) \\ &\qquad + (-\tfrac{7}{128})(-2135) + \tfrac{35}{2048}(-13) \\ &= 281\ 5737\ 9665 + 2277\ 50829\cdot75 - 14\ 3054\cdot34 \\ &\qquad + 116\cdot76 - 0\cdot22 \\ &= 283\ 8498\ 7556\cdot95. \end{aligned}$$

$\therefore$ log cosh $(0\cdot3655) = \underline{0\cdot0283\ 8498\ 7557}$.

By the *Newton-Stirling* formula

$$\begin{aligned} f(0\cdot3655) &= 281\ 5737\ 9665 + \tfrac{3}{4}\cdot\tfrac{1}{2}\begin{pmatrix} 3\ 0214\ 1860 \\ +\ 3\ 0366\ 7773 \end{pmatrix} + \tfrac{9}{32}(152\ 5913) \\ &\qquad + (-\tfrac{7}{128})\tfrac{1}{2}\begin{pmatrix} -2122 \\ -2135 \end{pmatrix} + (-\tfrac{21}{2048})(-13) \\ &= 281\ 5737\ 9665 + 2\ 2717\ 8612\cdot38 + 42\ 9163\cdot03 \\ &\qquad + 116\cdot40 + \cdot13 \\ &= 283\ 8498\ 7556\cdot94. \end{aligned}$$

$\therefore$ log cosh $(0\cdot3655) = \underline{0\cdot0283\ 8498\ 7557}$.

27. **The Formulae of the preceding Sections may be expressed more concisely by means of the Central-Difference Notation of § 20.**

Everett's formula:

$$\begin{aligned} u_x &= \xi u_0 + (\xi+1)_3\delta^2u_0 + (\xi+2)_5\delta^4u_0 + \ldots + (\xi+r)_{2r+1}\delta^{2r}u_0 + \ldots \\ &\quad + xu_1 + (x+1)_3\delta^2u_1 + (x+2)_5\delta^4u_1 + \ldots + (x+r)_{2r+1}\delta^{2r}u_1 + \ldots \end{aligned}$$

The *Newton-Bessel* formula:

$$\begin{aligned} u_x &= \mu u_{\frac{1}{2}} + (x-\tfrac{1}{2})\delta u_{\frac{1}{2}} + (x)_2\mu\delta^2u_{\frac{1}{2}} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\delta^3u_{\frac{1}{2}} \\ &\quad + (x+1)_4\mu\delta^4u_{\frac{1}{2}} + \frac{(x+1)x(x-1)(x-2)(x-\frac{1}{2})}{5!}\delta^5u_{\frac{1}{2}} + \ldots \\ &\quad + \ldots + (x+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}} + (x+r-1)_{2r}\frac{x-\frac{1}{2}}{2r+1}\delta^{2r+1}u_{\frac{1}{2}} + \ldots \end{aligned}$$

The *Newton-Gauss* formula:

$$\begin{aligned} u_x &= u_0 + x\delta u_{\frac{1}{2}} + (x)_2\delta^2u_0 + (x+1)_3\delta^3u_{\frac{1}{2}} + (x+1)_4\delta^4u_0 + (x+2)_5\delta^5u_{\frac{1}{2}} \\ &\qquad + \ldots + (x+r-1)_{2r}\delta^{2r}u_0 + (x+r)_{2r+1}\delta^{2r+1}u_{\frac{1}{2}} + \ldots \end{aligned}$$

The *Newton-Stirling* formula:

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1^2)}{4!}\delta^4 u_0$$
$$+ \frac{x(x^2-1^2)(x^2-2^2)}{5!}\mu\delta^5 u_0 + \frac{x^2(x^2-1^2)(x^2-2^2)}{6!}\delta^6 u_0 + \dots$$
$$+ \tfrac{1}{2}\{(x+r)_{2r} + (x+r-1)_{2r}\}\delta^{2r} u_0 + (x+r)_{2r+1}\mu\delta^{2r+1} u_0 + \dots$$

Newton-Gauss backward formula:

$$u_{-x} = u_0 - x\delta u_{-\frac{1}{2}} + (x)_2\delta^2 u_0 - (x+1)_3\delta^3 u_{-\frac{1}{2}}$$
$$+ (x+1)_4\delta^4 u_0 - (x+2)_5\delta^5 u_{-\frac{1}{2}} + \dots$$
$$+ (x+r-1)_{2r}\delta^{2r} u_0 - (x+r)_{2r+1}\delta^{2r+1} u_{-\frac{1}{2}} + \dots$$

28. **The Lozenge Diagram.**—We shall now give a method which enables us to find a large number of formulae of interpolation.

Let $(p)_q$ denote the quantity $\frac{p!}{q!(p-q)!}$, and let u_r denote the entry $f(a+rw)$. We obtain at once the relations

$$(p)_q = (p+1)_{q+1} - (p)_{q+1}, \qquad (1)$$
$$\Delta^q u_{-r+1} - \Delta^q u_{-r} = \Delta^{q+1} u_{-r}, \qquad (2)$$

and, combining these equations, we see that

$$(p)_q\{\Delta^q u_{-r+1} - \Delta^q u_{-r}\} = \{(p+1)_{q+1} - (p)_{q+1}\}\Delta^{q+1} u_{-r},$$

or

$$(p)_q\Delta^q u_{-r} + (p+1)_{q+1}\Delta^{q+1} u_{-r} = (p)_q\Delta^q u_{-r+1} + (p)_{q+1}\Delta^{q+1} u_{-r}. \qquad (3)$$

Suppose we arrange these terms in the form of a "lozenge" so that the terms on the left-hand side of the equation lie along the two upper sides of the lozenge and the terms of the right-

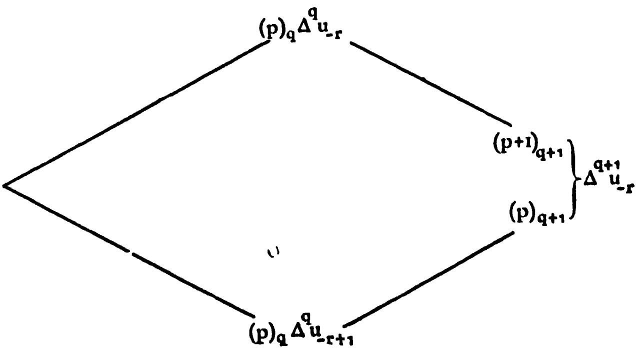

FIG. 2.

hand side along the lower sides. We obtain the above diagram in which a line directed from left to right joining two quantities denotes the *addition* of those quantities.

Equation (3) may be expressed by the statement that: *in travelling from the left-hand vertex to the right-hand vertex of the lozenge in the diagram, the sum of the elements which lie along the upper route is equal to the sum of the elements which lie along the lower route.*

It is evident that this statement may be extended. For example, let us place in contiguity the lozenges corresponding to

$$\begin{pmatrix} p=n \\ q=1 \\ r=1 \end{pmatrix} \begin{pmatrix} p=n-1 \\ q=1 \\ r=0 \end{pmatrix} \begin{pmatrix} p=n \\ q=2 \\ r=1 \end{pmatrix}$$

so that the upper vertices of the lozenges, which are of the form $(p)_q\Delta^q u_{-r}$, form a sort of difference table:

$$\begin{array}{ll} (n)_1\Delta u_{-1} & \\ & (n)_2\Delta^2 u_{-1} \\ (n-1)_1\Delta u_0 & \end{array}$$

We obtain the following diagram:

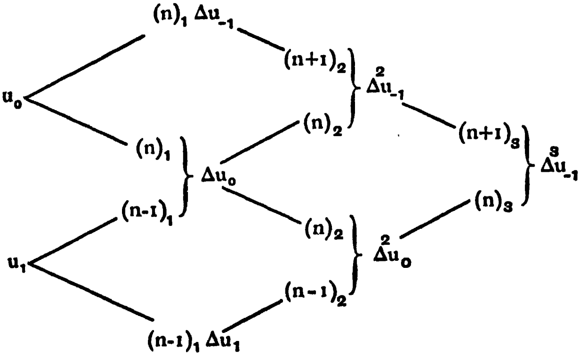

FIG. 3.

Applying the rule given by equation (3), it is evident that the sum of the elements along either of the following routes is the same:

$$\begin{aligned} &u_0+(n)_1\Delta u_{-1}+(n+1)_2\Delta^2 u_{-1}+(n+1)_3\Delta^3 u_{-1}, \\ &u_0+(n)_1\Delta u_0 \quad +(n)_2\Delta^2 u_{-1} \qquad +(n+1)_3\Delta^3 u_{-1}, \\ &u_0+(n)_1\Delta u_0 \quad +(n)_2\Delta^2 u_0 \qquad\;\; +(n)_3\Delta^3 u_{-1}. \end{aligned}$$

Since $u_0 + (n)_1\Delta u_0 = u_1 + (n-1)_1\Delta u_0$, we may form three other expressions beginning with the term u_1 instead of u_0 and equivalent to those already given, namely,

$$u_1 + (n-1)_1\Delta u_0 + (n)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-1}$$

and two similar expressions.

If we examine the structure of this diagram, it will be seen that the values of q and r in the expression $(p)_q\Delta^q u_{-r}$ are arranged in precisely the same way as for the differences $\Delta^q f(u - rw)$ in an ordinary difference table. The values of p are constant along any diagonal descending from left to right of the diagram, while along a diagonal ascending from left to right these values increase by unity at each vertex. The first value of p along either line radiating from u_0 is taken to be $p = n$.

By extending this diagram we arrive at the following, which

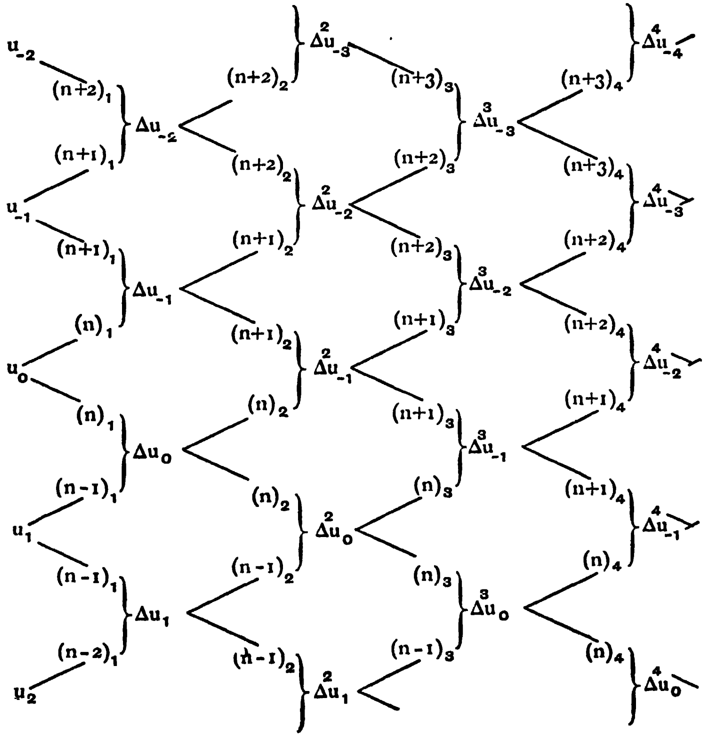

FIG. 4.

may be called a *lozenge* or "*Fraser*" diagram since it is a modification of one due to D. C. Fraser.*

Now the Gregory-Newton formula for u_n is the sum of the elements from u_0 along the downward sloping line to the line of zero differences. So $u_n =$ the sum of the elements from u_0 *along any route whatever* to the line of zero differences.

From the identity $u_0 + n\Delta u_0 = u_1 + (n-1)\Delta u_0$ it is evident that the value of u_n is unaltered if a route is selected starting from u_1 instead of from u_0. *In general the sum of the elements along any route proceeding from any entry* u_r *whatever to the line of zero differences is equal to* u_n.

Applying this rule, we have at once from the lozenge diagram

$$u_n = u_0 + (n)_1\Delta u_{-1} + (n+1)_2\Delta^2 u_{-2} + (n+2)_3\Delta^3 u_{-3} + (n+3)_4\Delta^4 u_{-4} + \dots \quad (4)$$

$$u_n = u_0 + (n)_1\Delta u_{-1} + (n+1)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-2} + (n+2)_4\Delta^4 u_{-2} + \dots \quad (5)$$

$$u_n = u_0 + (n)_1\Delta u_0 + (n)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-1} + (n+1)_4\Delta^4 u_{-2} + \dots \quad (6)$$

$$u_n = u_1 + (n-1)_1\Delta u_0 + (n)_2\Delta^2 u_0 + (n)_3\Delta^3 u_{-1} + (n+1)_4\Delta^4 u_{-1} + \dots \quad (7)$$

Rewriting equations (5), (6) in the central-difference notation, we find

$$u_n = u_0 + (n)_1\delta u_{-\frac{1}{2}} + (n+1)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{-\frac{1}{2}} + (n+2)_4\delta^4 u_0 + \dots$$

and

$$u_n = u_0 + (n)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_0 + \dots$$

which is the *Newton-Gauss* formula.

If we now take the mean of these values of u_n, we obtain the formula whose differences are along the row corresponding to u_0:

$$u_n = u_0 + (n)_1\tfrac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}}) + \tfrac{1}{2}\{(n+1)_2 + (n)_2\}\delta^2 u_0 + (n+1)_3\tfrac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}}) + \tfrac{1}{2}\{(n+2)_4 + (n+1)_4\}\delta^4 u_0 + \dots$$

or

$$u_n = u_0 + (n)_1\mu\delta u_0 + \tfrac{1}{2}n^2\delta^2 u_0 + \tfrac{1}{6}n(n^2-1)\mu\delta^3 u_0 + \tfrac{1}{24}n^2(n^2-1)\delta^4 u_0 + \dots$$

which is the *Newton-Stirling* formula.

The mean value of u_n from equations (6), (7) may be expressed either as Everett's formula or as the Newton-Bessel formula. Writing (6), (7) in the central-difference notation,

$$u_n = u_0 + (n)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_0 + \dots + (n+r-1)_{2r}\delta^{2r} u_0 + (n+r)_{2r+1}\delta^{2r+1} u_{\frac{1}{2}} + \dots \quad (8)$$

$$u_n = u_1 + (n-1)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_1 + (n)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_1 + \dots + (n+r-1)_{2r}\delta^{2r} u_1 + (n+r-1)_{2r+1}\delta^{2r+1} u_{\frac{1}{2}} + \dots \quad (9)$$

Taking the arithmetic mean of these values of u_n, we may eliminate differences of *odd* order by applying the relations $(p)_q = (p+1)_{q+1} - (p)_{q+1}$ and $\delta^{2r+1}u_{\frac{1}{2}} = \delta^{2r}u_1 - \delta^{2r}u_0$. The coefficient of $\delta^{2r}u_1$ takes the form $\frac{1}{2}\{(n+r-1)_{2r} + (n+r)_{2r+1} + (n+r-1)_{2r+1}\}$ or $(n+r)_{2r+1}$. The coefficient of $\delta^{2r}u_0$ becomes $\frac{1}{2}\{(n+r-1)_{2r} - (n+r)_{2r+1} - (n+r-1)_{2r+1}\}$ or $-(n+r-1)_{2r+1}$, and by substituting ξ for $(1-n)$ we see that

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* *J.I.A.* **43** (1909), p. 238.

$$-(n+r-1)_{2r+1} = -(r-\xi)_{2r+1} = (\xi+r)_{2r+1}.$$

The arithmetic mean of equations (8), (9) may thus be written in the form

$$\begin{aligned} u_n = {} & \xi u_0 + (\xi+1)_3\delta^2 u_0 + (\xi+2)_5\delta^4 u_0 + \ldots + (\xi+r)_{2r+1}\delta^{2r}u_0 + \ldots \\ & + nu_1 + (n+1)_3\delta^2 u_1 + (n+2)_5\delta^4 u_1 + \ldots + (n+r)_{2r+1}\delta^{2r}u_1 + \ldots \end{aligned}$$

which is *Everett's* formula.

Suppose, however, we find the arithmetic mean of the values of u_n in (8) and (9) and simplify the coefficients of differences of odd order in the resulting expression by means of the relation

$$\tfrac{1}{2}\{(n+r)_{2r+1} + (n+r-1)_{2r+1}\} = (n+r-1)_{2r} \cdot \frac{n-\frac{1}{2}}{2r+1}.$$

We now obtain the result

$$\begin{aligned} u_n = {} & \mu u_{\frac{1}{2}} + (n-\tfrac{1}{2})\delta u_{\frac{1}{2}} + (n)_2\mu\delta^2 u_{\frac{1}{2}} + \frac{n(n-1)(n-\frac{1}{2})}{3!}\delta^3 u_{\frac{1}{2}} + (n+1)_4\mu\delta^4 u_{\frac{1}{2}} + \ldots \\ & + (n+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}} + (n+r-1)_{2r}\frac{n-\frac{1}{2}}{2r+1}\delta^{2r+1}u_{\frac{1}{2}} + \ldots \end{aligned}$$

which is the *Newton-Bessel* formula.

29. Relative Accuracy of Central-Difference Formulae.—It is frequently necessary to use approximate formulae which terminate before the column of zero differences is reached. From the last section we have seen that the sums of the elements along any two routes which terminate at the same vertex are identical. If the routes terminate at two adjacent vertices $(p)_q\Delta^q u_{-r+1}$ and $(p)_q\Delta^q u_{-r}$ which are in the same "lozenge," the sums of the elements along these routes differ by $(p)_q(\Delta^q u_{-r+1} - \Delta^q u_{-r})$, *i.e.* by $(p)_q\Delta^{q+1}u_{-r}$. Extending this result to routes terminating in the same column of differences, for example, at $\Delta^4 u_{-3}$ and $\Delta^4 u_0$, it is evident that the sums of the elements along these routes differ by $(n+2)_4\Delta^5 u_{-3} + (n+1)_4\Delta^5 u_{-2} + (n)_4\Delta^5 u_{-1}$.

We shall now consider routes that lie along horizontal lines; these yield the formulae containing mean-differences. In the last section it was shown that a mean-difference formula is obtained by taking the arithmetic mean of the elements along two adjacent routes. From the mode of formation we see that the sums of the elements along such routes are identical as far as the vertices at the intersections of the routes. For example, the Newton-Gauss formula is equivalent to the Newton-Stirling formula as far as differences of *even* order, and it is also equivalent to the Newton-Bessel formula as far as differences of *odd* order. When a formula is curtailed, the question arises as to whether it is more advantageous to select a route which terminates at a mean difference or at an ordinary difference.

The following diagram represents the portion of the lozenge diagram along the row corresponding to u_0 and adjacent to the differences of order $2r$. Let A denote the mean difference $(n+r)_{2r+1}\mu\delta^{2r+1}u_0$, and let B denote the mean difference $(n+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}}$.

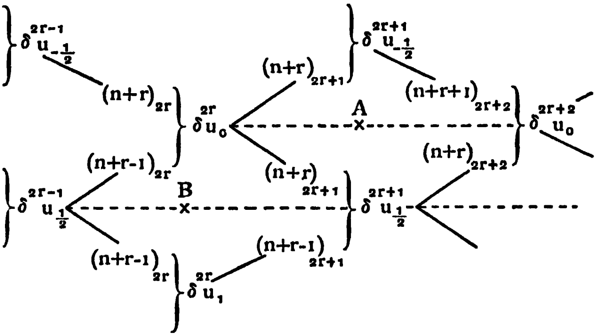

FIG. 5.

The route along the dotted line through A represents the *Newton-Stirling* formula and the route along the dotted line through B represents the *Newton-Bessel* formula. The *Newton-Gauss* formula, which is represented in the diagram by a zigzag intermediate route, is equivalent to the Stirling formula at the vertices $\delta^{2r}u_0$ and $\delta^{2r+2}u_0$, and it is also equivalent to the Bessel formula at the vertices $\delta^{2r-1}u_{\frac{1}{2}}$ and $\delta^{2r+1}u_{\frac{1}{2}}$.

Consider the three routes representing the Gauss and the Stirling formulae and the formula which contains the differences $\delta^{2r-1}u_{-\frac{1}{2}}$, $\delta^{2r}u_0$, $\delta^{2r+1}u_{-\frac{1}{2}}$, and $\delta^{2r+2}u_0$. If we suppose these formulae to be curtailed so that the last difference of each is of order $2r+1$, we may compare the accuracy of these formulae by ascertaining the magnitude of the neglected terms of order $(2r+2)$. The sum of the elements along either of the routes from the common vertex $\delta^{2r+2}u_0$ to the line of zero differences being the same, the most accurate formula is the one in which the neglected term of order $(2r+2)$ is the smallest. These terms are:

$$(n+r)_{2r+2}\delta^{2r+2}u_0, \quad \tfrac{1}{2}\{(n+r)_{2r+2}+(n+r+1)_{2r+2}\}\delta^{2r+2}u_0, \quad (n+r+1)_{2r+2}\delta^{2r+2}u_0$$

respectively, and they are also arranged in ascending order of magnitude. The Newton-Gauss formula is therefore more accurate as far as mean differences of order $(2r+1)$, when further terms are neglected, than the corresponding Newton-Stirling formula passing through the same differences of even order; and both are more accurate than the formula containing the difference $\delta^{2r+1}u_{-\frac{1}{2}}$. In precisely the same way we see that the Bessel formula is more accurate than the Gauss formula as far as differences of even order when further terms are neglected. In general, *a central-difference formula terminating at a mean difference of the entry u_p is more accurate than a formula which is curtailed at the corresponding central-difference of $u_{p-\frac{1}{2}}$, and it is less accurate than a formula which is curtailed at the corresponding difference of $u_{p+\frac{1}{2}}$.**

We shall now illustrate by an example the superiority which central-difference formulae generally have over other interpolation formulae.

Let it be required to find u_x, where $-\frac{1}{2}<x<\frac{1}{2}$. If we employ for this purpose an interpolation formula which proceeds according to central differences of u_0, and stop at the $(2r+1)$th term, the result is the same as if we employed Lagrange's formula with given values of $u_{-r}, u_{-r+1}, \ldots, u_r$, so that by § 19 the error is

$$\frac{(x+r)(x+r-1)\ .\ .\ .\ (x-r)}{(2r+1)!}f^{(2r+1)}(\xi),$$

where ξ denotes some number between $a-rw$ and $a+rw$. If, on the other hand, we employ the Gregory-Newton formula, and stop at the $(2r+1)$th term, the result we thereby obtain is the same as if we employed Lagrange's formula with given values of $u_0, u_1, \ldots, u_{2r}$, so that the error is

$$\frac{x(x-1)\ .\ .\ .\ (x-2r)}{(2r+1)!}f^{(2r+1)}(\eta),$$

where η denotes some number between a and $a+2rw$. Now $f^{(2r+1)}(\xi)$ does not, in most cases, differ greatly from $f^{(2r+1)}(\eta)$, but $(x+r)(x+r-1)$ $\ldots (x-r)$ is much smaller than $x(x-1) \ldots (x-2r)$ in absolute value when $-\frac{1}{2}<x<\frac{1}{2}$. Thus the error is smaller in the former case than in the latter. For this reason central-difference formulae are preferable to the ordinary formulae for advancing differences.

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* A detailed discussion of the accuracy of interpolation formulae is given in papers by W. F. Sheppard, *Proc. Lond. Math. Soc.* **4** (1906), p. 320, and **10** (1911), p. 139; D. C. Fraser, *J.I.A.* **50**, pp. 25-27; G. J. Lidstone, *Trans. Fac. Act.* **9** (1923).

The following remarks* are of general application:

"Formulas which proceed to constant differences are exact, and are true for all values of n whether integral or fractional.

"Formulas which stop short of constant differences are approximations.

"Approximate formulas which terminate with the same difference are identically equal.

"Approximate formulas which terminate with distinct differences of the same order are not identical. The difference between them is expressed by the chain of lines necessary to complete the circuit."

30. Preliminary Transformations.

—In certain cases formulae of interpolation should not be used until some preliminary transformation has been effected. We shall illustrate this by two examples.

Ex. 1.—Suppose that it is required to find L sin 15″. We have from a table of logarithms the following entries:

θ.	*L sin* θ.			
0° 0′ 10″	5·6855749			
		3010300		
20″	5·9866049		−1249388	
		1760912		737864
30″	6·1626961		−511524	
		1249388		231236
40″	6·2876349		−280288	
		969100		
50″	6·3845449			

The differences are evidently very slowly convergent. One reason for this will be seen when it is remembered that when θ is small and $\theta'' = x$ radians, then $\sin x = x - \frac{1}{6}x^3 + \ldots$ and $x = \theta \sin 1''$ (nearly), so that $\mathrm{L} \sin \theta = \mathrm{L} \sin 1'' + \log \theta$ (nearly), and the differences of $\log \theta$ for the values 10, 20, 30, 40, 50 . . . of θ are very slowly convergent. We therefore calculate L sin θ when θ is small by adding the interpolated values of $\mathrm{L}\left(\frac{\sin \theta}{\theta}\right)$, which has regular differences, and $\log \theta$, for which tables exist with smaller intervals of the argument.

Ex. 2.—Suppose it is required to interpolate between two terms of such a sequence as the following:

$$1, \frac{r}{p}, \frac{r(r+1)}{p(p+1)}, \frac{r(r+1)(r+2)}{p(p+1)(p+2)}, \frac{r(r+1)(r+2)(r+3)}{p(p+1)(p+2)(p+3)}, \ldots$$

where r and p are two widely different numbers.

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* D. C. Fraser, *J.I.A.* 43 (1909), p. 238.

It is best to interpolate in the sequence of numerators

$$1,\ r,\ r(r+1),\ r(r+1)(r+2),\ \ldots$$

and to interpolate separately in the sequence of denominators

$$1,\ p,\ p(p+1),\ p(p+1)(p+2)\ \ldots$$

We then divide the former result by the latter, in order to obtain the required interpolated value.

Stirling (*Methodus Differentialis* (1730), Prop. xvii. Scholium) says: "As in common algebra the whole art of the analyst does not consist in the resolution of the equations but in bringing the problems thereto; so likewise in this analysis: there is less dexterity required in the performance of the process of interpolation than in the preliminary determination of the sequences which are best fitted for interpolation."

The general rule is to make such transformations as will make the interpolation as simple as possible.

EXAMPLES ON CHAPTER III

1. Given

$$\begin{aligned} \sin 25^\circ\, 41'\, 40'' &= 0{\cdot}433\ 571\ 711\ 655\ 565 \\ \sin 25^\circ\, 42'\ \ 0'' &= 0{\cdot}433\ 659\ 084\ 587\ 544 \\ 20'' &= 0{\cdot}433\ 746\ 453\ 442\ 359 \\ 40'' &= 0{\cdot}433\ 833\ 818\ 219\ 189 \end{aligned}$$

find the value of $\sin 25^\circ\, 42'\, 10''$ by the Newton-Gauss formula.

2. Find the value of $\log \sin 0^\circ\, 16'\, 8''{\cdot}5$ having given

$$\begin{aligned} \log \sin 0^\circ\, 16'\, 7'' &= 7{\cdot}670\ 999\ 750\ 0 \\ 8'' &= 7{\cdot}671\ 448\ 629\ 9 \\ 9'' &= 7{\cdot}671\ 897\ 046\ 4 \\ 10'' &= 7{\cdot}672\ 345\ 000\ 2 \end{aligned}$$

using the Newton-Gauss formula.

Check your result by obtaining $\log \sin 0^\circ\, 16'\, 8''{\cdot}5$ from the following data:

$$\begin{aligned} \log \sin 0^\circ\, 16'\, 6'' &= 7{\cdot}670\ 550\ 405\ 5 \\ 8'' &= 7{\cdot}671\ 448\ 629\ 9 \\ 10'' &= 7{\cdot}672\ 345\ 000\ 2 \\ 12'' &= 7{\cdot}673\ 239\ 524\ 3 \end{aligned}$$

3. Apply the Newton-Stirling formula to compute $\sin 25^\circ\, 40'\, 30''$ from the table of values

$$\begin{aligned} \sin 25^\circ\, 40'\, 0'' &= 0{\cdot}433134785866963 \\ 20'' &= 0{\cdot}433222179172439 \\ 40'' &= 0{\cdot}433309568404859 \\ \sin 25^\circ\, 41'\, 0'' &= 0{\cdot}433396953563401 \\ 20'' &= 0{\cdot}433484334647243 \end{aligned}$$

and verify your answer, using the Newton-Bessel formula.

4. Given

$$\begin{aligned}\log 310 &= 2{\cdot}4913617\\ 320 &= 2{\cdot}5051500\\ 330 &= 2{\cdot}5185139\\ 340 &= 2{\cdot}5314789\\ 350 &= 2{\cdot}5440680\\ 360 &= 2{\cdot}5563025\end{aligned}$$

find the value of log 3375 by the Newton-Bessel formula, verifying the result by one or more other central-difference formulae and comparing it with the true value. [3·5282738.]

5. Show that the lozenge-diagram method really derives all the interpolation formulae by repeated summation by parts, *i.e.* by the use of the formula

$$u_{x+1}\Delta v_x = \Delta(u_x v_x) - v_x \Delta u_x,$$

which is the analogue in the Calculus of Differences of the formula

$$\int u\,dv = uv - \int v\,du$$

in the Integral Calculus.

CHAPTER IV

APPLICATIONS OF DIFFERENCE FORMULAE

31. **Subtabulation.**—An important application of interpolation formulae is to the extension of tables of a function. Thus, supposing we already possess a table giving $\sin x$ at intervals of $1'$ of x, we might wish to construct a table giving $\sin x$ at intervals of $10''$ of x. This operation is called *subtabulation.* Subtabulation might evidently be performed by calculating each of the new values by ordinary interpolation, but when the new values are required in this wholesale fashion it is better to proceed otherwise, forming first the *differences* of the new sequence of values of the function, and then calculating the latter from those differences.*

Let $T_0, T_1, T_2, T_3, \ldots$ be a given sequence of entries in a table corresponding to intervals w of the argument, and let their successive differences be $\Delta T_0 = T_1 - T_0$, $\Delta^2 T_0 = T_2 - 2T_1 + T_0$, etc. Suppose it is desired to find the values of the function in question at intervals w/m of the argument so that $(m-1)$ intermediate values are to be interpolated between every two consecutive members of the set $T_0, T_1, T_2 \ldots$ Denote the sequences thus required by $t_0, t_1, t_2, \ldots$, so that $t_0 = T_0$, $t_m = T_1$, $t_{2m} = T_2$, $t_{3m} = T_3$, etc., and let the successive differences in the new sequence be

$$\Delta_1 t_0 = t_1 - t_0, \quad \Delta_1{}^2 t_0 = t_2 - 2t_1 + t_0, \text{ etc.},$$

where Δ_1 is used instead of Δ to denote the operation of differencing in the new sequence. The differences in the new sequence may now be found in terms of the differences in the old sequence by the use of operators in the following way.

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* Lagrange, *Œuvres*, **5**, p. 663 (1792–3). Cf. L. J. Comrie, *Monthly Notices R.A.S.*, **88** (1928), p. 506. F. Emde, *Zeitschr. f. angew. Math.*, **14** (1934) 333. K. Camp, *Trans. Act. Soc. Amer.*, **38** (1937) 16. The discovery of subtabulation should be credited to H. Briggs (1561–1631), who used it in his computation of log. tables.

Denoting the initial value t_0 or T_0 by $f(a)$, we have by the Gregory-Newton interpolation formula:

$$t_1 = f(a + w/m) = T_0 + (1/m)_1 \Delta T_0 + (1/m)_2 \Delta^2 T_0 + (1/m)_3 \Delta^3 T_0 + \ldots$$

and the operators Δ_1 and Δ are thus connected by the relation

$$\Delta_1 = (1/m)_1 \Delta + (1/m)_2 \Delta^2 + (1/m)_3 \Delta^3 + \ldots \qquad (1)$$

Suppose for simplicity that $\Delta^4 T_0$ is the last non-zero difference of the original sequence, so that $\Delta^5 T_0 = 0$, $\Delta^6 T_0 = 0$, etc. Equation (1) gives

$$\Delta_1^s = \{(1/m)_1 \Delta + (1/m)_2 \Delta^2 + (1/m)_3 \Delta^3 + (1/m)_4 \Delta^4\}^s. \qquad (2)$$

If we now substitute the values $s = 1, 2, 3, 4$ in the last equation, we are able to determine all the differences of the new sequence in terms of the differences of the old sequence:

$$\Delta_1 t_0 = \frac{1}{m}\Delta T_0 + \frac{1-m}{2m^2}\Delta^2 T_0 + \frac{(1-m)(1-2m)}{6m^3}\Delta^3 T_0 + \frac{(1-m)(1-2m)(1-3m)}{24m^4}\Delta^4 T_0. \qquad (3)$$

$$\Delta_1^2 t_0 = \frac{1}{m^2}\Delta^2 T_0 + \frac{1-m}{m^3}\Delta^3 T_0 + \frac{(1-m)(7-11m)}{12m^4}\Delta^4 T_0, \qquad (4)$$

$$\Delta_1^3 t_0 = \frac{1}{m^3}\Delta^3 T_0 + \frac{3(1-m)}{2m^4}\Delta^4 T_0, \qquad (5)$$

$$\Delta_1^4 t_0 = \frac{1}{m^4}\Delta^4 T_0. \qquad (6)$$

When the differences are thus calculated, the entries t_1, t_2, t_3 may be derived in the usual way by simple addition. The values of t_m, t_{2m}, t_{3m}, . . . formed in this way should agree with the tabulated values T_1, T_2, T_3, . . .

Ex.—The logs of the numbers 1500, 1510, 1520, 1530, 1540 *being given to nine places of decimals, to find the logs of the integers between* 1500 *and* 1510.

The difference table of the original values is as follows:

No.	*log.*	Δ.	Δ^2.	Δ^3.	Δ^4.
1500	176091259				
		2885688			
1510	178976947		− 19047		
		2866641		249	
1520	181843588		− 18798		− 4
		2847843		245	
1530	184691431		− 18553		
		2829290			
1540	187520721				

Here $m = 10$.

$\therefore\ \Delta_1^4 = \frac{1}{10^4}(-4) = -0{\cdot}0004$ in the ninth place, which is negligible,

$\Delta_1^3 = \frac{1}{10^3}249 + \frac{3(1-10)}{2\cdot 10^4}(-4) = 0{\cdot}2544 = 0{\cdot}25$

which is approximately constant,

$\Delta_1^2 = \frac{1}{10^2}(-19047) + \frac{(1-10)}{10^3}249 + \frac{(1-10)(7-110)}{12\cdot 10^4}(-4) = -192{\cdot}74,$

$$\Delta_1 = \frac{1}{10}2885688 + \frac{(1-10)}{2\cdot 10^2}(-19047) + \frac{(1-10)(1-20)}{6\cdot 10^3}249 + \frac{(1-10)(1-20)(1-30)}{24\cdot 10^4}(-4)$$

$$\begin{aligned} &= 288568{\cdot}8 \\ &\quad\ \ 857{\cdot}115 \\ &\quad\ \ \ \ 7{\cdot}0965 \\ &\quad\ \ \ \ 0{\cdot}08265 \\ &= 289433{\cdot}094 \end{aligned}$$

No.	*logs.*	Δ_1.	Δ_1^2.	Δ_1^3.
1500	176091259·1			
		289433·1		
1501	176380692·2		− 192·74	
		289240·4		0·25
1502	176669932·6		− 192·49	
		289047·9		0·25
1503	176958980·5		− 192·24	
		288855·6		0·25
1504	177247836·1		− 191·99	
		288663·6		0·25
1505	177536499·7		− 191·74	
		288471·9		0·25
1506	177824971·6		− 191·49	
		288280·4		0·25
1507	178113252·0		− 191·24	
		288089·2		0·25
1508	178401341·2		− 190·99	
		287898·2		0·25
1509	178689239·4		− 190·74	
		287707·4		
1510	178976946·8			

The required new table is :

No.	*log.*	*No.*	*log.*
1500	3·176091259	1506	3·177824972
1501	3·176380692	1507	3·178113252
1502	3·176669933	1508	3·178401341
1503	3·176958981	1509	3·178689239
1504	3·177247836	1510	3·178976947
1505	3·177536500		

and the final value of log 1510 agrees with the original value.

32. An Alternative Derivation.—It is frequently convenient when dealing with a function whose degree is known to insert values of the function, intermediate to those already tabulated, by the following method:

Suppose, for example, that a function $f(x)$ may be represented by a polynomial of the third degree, and that values of the function are tabulated at intervals $w=10$ of the argument. Let it be required to insert values at an interval $w=1$. Using the notation of the last section, we have (by the Gregory-Newton formula)

$$\begin{aligned}
T_0 &= t_0,\\
T_1 &= t_{10} = t_0 + 10\Delta_1 t_0 + 45\Delta_1{}^2 t_0 + 120\Delta_1{}^3 t_0,\\
T_2 &= t_{20} = t_0 + 20\Delta_1 t_0 + 190\Delta_1{}^2 t_0 + 1140\Delta_1{}^3 t_0,\\
T_3 &= t_{30} = t_0 + 30\Delta_1 t_0 + 435\Delta_1{}^2 t_0 + 4060\Delta_1{}^3 t_0.
\end{aligned}$$

Differencing these equations, we see that

$$\begin{aligned}
\Delta T_0 &= 10\Delta_1 t_0 + 45\Delta_1{}^2 t_0 + 120\Delta_1{}^3 t_0,\\
\Delta T_1 &= 10\Delta_1 t_0 + 145\Delta_1{}^2 t_0 + 1020\Delta_1{}^3 t_0,\\
\Delta T_2 &= 10\Delta_1 t_0 + 245\Delta_1{}^2 t_0 + 2920\Delta_1{}^3 t_0.
\end{aligned}$$

Similarly

$$\begin{aligned}
\Delta^2 T_0 &= 100\Delta_1{}^2 t_0 + 900\Delta_1{}^3 t_0,\\
\Delta^2 T_1 &= 100\Delta_1{}^2 t_0 + 1900\Delta_1{}^3 t_0,\\
\therefore\ \Delta^3 T_0 &= 1000\Delta_1{}^3 t_0.
\end{aligned}$$

The leading term and its differences for the subdivided intervals are seen to be

$$\begin{aligned}
\Delta_1{}^3 t_0 &= \cdot 001\Delta^3 T_0,\\
\Delta_1{}^2 t_0 &= \cdot 01\Delta^2 T_0 - \cdot 009\Delta^3 T_0,\\
\Delta_1 t_0 &= \cdot 1\Delta T_0 - \cdot 045\Delta^2 T_0 + \cdot 0285\Delta^3 T_0,^{*}
\end{aligned}$$

from which the values $t_1, t_2, t_3, \ldots$ are formed by addition.

Ex.—Having given a table of values of log x at intervals of the argument $w=5$, to insert between log 6250 and log 6255 the intermediate values of the function at intervals $w=1$.

	Entry.	Δ.	Δ^2.
Put	$T_0 = \log 6250 = 3\cdot 7958800$		
		3473	
	$T_1 = \log 6255 = 3\cdot 7962273$		-3
		3470	
	$T_2 = \log 6260 = 3\cdot 7965743$		-2
		3468	
	$T_3 = \log 6265 = 3\cdot 7969211$		

The differences of the second order are approximately constant, so we assume log x to be a polynomial of the second degree.

$$\begin{aligned}
T_0 &= t_0 = 3\cdot 7958800,\\
T_1 &= t_5 = t_0 + 5\Delta_1 t_0 + 10\Delta_1{}^2 t_0,\\
T_2 &= t_{10} = t_0 + 10\Delta_1 t_0 + 45\Delta_1{}^2 t_0.\\
\Delta T_0 &= 5\Delta_1 t_0 + 10\Delta_1{}^2 t_0 = 3473,\\
\Delta T_1 &= 5\Delta_1 t_0 + 35\Delta_1{}^2 t_0.\\
\Delta^2 T_0 &= 25\Delta_1{}^2 t_0 = -3.
\end{aligned}$$

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* These are precisely the set of equations of § 31 when $\Delta^3 T_0$, the third differences of the tabulated function, are assumed to be constant.

From these equations we obtain the values

$$\Delta_1{}^2 t_0 = -0{\cdot}12, \quad \Delta_1 t_0 = 694{\cdot}84,$$

expressed in units of the seventh decimal place.

Forming the difference table for the subdivided intervals,

Entry.	Δ_1.	$\Delta_1{}^2$.
log 62 50 = 37958800·00		
	694·84	
log 62 51 = 37959494·84		−0·12
	694·72	
log 62 52 = 37960189·56		−0·12
	694·60	
log 62 53 = 37960884·16		−0·12
	694·48	
log 62 54 = 37961578·64		−0·12
	694·36	
log 62 55 = 37962273·00		

We may now insert these values of the function in the table of values, thus:

$$\log 6251 = 3{\cdot}7959495$$
$$\log 6252 = 3{\cdot}7960190, \text{ etc.}$$

We may obtain without difficulty formulae for subtabulation based on central-difference formulae, or on Everett's formula. These are frequently to be preferred to the subtabulation formulae based on the Gregory-Newton formula.

Owing to the rapid accumulation of error in the higher orders of differences, care must be taken to include additional places of digits in the computations, as in the above examples.

33. Estimation of Population for Individual Ages when Populations are given in Age Groups.—We shall now find the values of a statistical quantity, such as the population of a given district, for individual years, when the sums of its values for quinquennial periods are given.*

Let . . ., u_{-2}, u_{-1}, u_0, u_1, u_2, . . . be the values of the quantity for individual years, and let the quinquennial sums be . . ., W_1, W_0, W_{-1}, . . . , so that

$$\begin{aligned} W_1 &= u_7 + u_6 + u_5 + u_4 + u_3, \\ W_0 &= u_2 + u_1 + u_0 + u_{-1} + u_{-2}, \\ W_{-1} &= u_{-3} + u_{-4} + u_{-5} + u_{-6} + u_{-7}. \end{aligned}$$

It is required to find the value u_0 in terms of the W's.

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* G. King, *J.I.A.* **43**, p. 109 (1909). See also **50**, p. 32.

The Newton-Stirling formula may be written

$$u_n = u_0 + n\frac{\Delta u_{-1} + \Delta u_0}{2} + \frac{n^2}{2}\Delta^2 u_{-1} + \frac{n(n^2-1)}{6}\,\frac{\Delta^3 u_{-2} + \Delta^3 u_{-1}}{2}$$
$$+ \frac{n^2(n^2-1)}{24}\Delta^4 u_{-2} + \ldots$$

If we denote $u_n + u_{-n}$ by y_n and neglect the differences of the fourth and higher orders, we may write

$$y_n = 2u_0 + n^2\Delta^2 u_{-1}.$$

Therefore
$$W_0 = u_0 + y_1 + y_2$$
$$= 5u_0 + 5\Delta^2 u_{-1},$$

and
$$W_1 + W_{-1} = y_3 + y_4 + y_5 + y_6 + y_7$$
$$= 10u_0 + 135\Delta^2 u_{-1}.$$

Eliminating $\Delta^2 u_{-1}$ from the two last equations, u_0 may be expressed in terms of the W's:

$$125u_0 = 27W_0 - (W_{-1} + W_1),$$

or, writing $\Delta^2 W_{-1}$ for $(W_{-1} - 2W_0 + W_1)$, we obtain the result

$$125u_0 = 25W_0 - \Delta^2 W_{-1},$$

or
$$u_0 = 0{\cdot}2W_0 - 0{\cdot}008\Delta^2 W_{-1}. \qquad (1)$$

Ex.—To find the value of the quantity for the middle year of the second quinquennium, when the following are three consecutive quinquennial sums: 36556 : 39387 : 41921.

Denote the given quinquennial sums by W_{-1}, W_0, W_1 respectively, and form a difference table.

$$\begin{array}{lcc}
W_{-1} = 36\,556 & & \\
 & 2831 & \\
W_0 = 39\,387 & & -297 \\
 & 2534 & \\
W_1 = 41\,921 & &
\end{array}$$

The required quantity u_0 is therefore, by (1),

$$u_0 = 0{\cdot}2 \times 39\,387 - {\cdot}008\,(-297)$$
$$= 7877{\cdot}4 + 2{\cdot}4$$
$$= 7879{\cdot}8,$$

so
$$u_0 = 7880.$$

The above formula may be extended to include the fourth differences of the W's when we neglect the differences of the u's of the sixth and higher orders.* We have now

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* When the groups are unequal, we can proceed in a similar way, using divided differences.

$$\begin{aligned}y_n&=u_n+u_{-n}\\&=2u_0+n^2\Delta^2u_{-1}+\tfrac{1}{12}n^2(n^2-1)\Delta^4u_{-2},\\ \mathrm{W}_0&=u_0+y_1+y_2\\&=5u_0+5\Delta^2u_{-1}+\Delta^4u_{-2},\qquad(2)\\ \mathrm{W}_1+\mathrm{W}_{-1}&=10u_0+135\Delta^2u_{-1}+377\Delta^4u_{-2},\\ \mathrm{W}_2+\mathrm{W}_{-2}&=10u_0+510\Delta^2u_{-1}+4627\Delta^4u_{-2}.\end{aligned}$$

Eliminating u_0 from the three last equations, we have

$$\Delta^2\mathrm{W}_{-1}=125\Delta^2u_{-1}+375\Delta^4u_{-2},$$

and $$\Delta\mathrm{W}_1-\Delta\mathrm{W}_{-2}=375\Delta^2u_{-1}+4250\Delta^4u_{-2},$$

and eliminating Δ_2u_{-1} from these two equations we find that

$$\Delta^4u_{-2}=0{\cdot}00032\Delta^4\mathrm{W}_{-2}$$

and $$\Delta^2u_{-1}=0{\cdot}008\Delta^2\mathrm{W}_{-1}-0{\cdot}00096\Delta^4\mathrm{W}_{-2}.$$

If we now substitute these values in equation (2), we obtain the result

$$u_0=0{\cdot}2(\mathrm{W}_0-5\Delta^2u_{-1}-\Delta^4u_{-2}),$$

or $$u_0=0{\cdot}2\mathrm{W}_0-0{\cdot}008\Delta^2\mathrm{W}_{-1}+0{\cdot}000896\Delta^4\mathrm{W}_{-2}.$$

This value of u_0 was also given by G. King.*

The following demonstration of a more general formula is due to G. J. Lidstone.

Let $$\mathrm{W}_0=\sum_{-r}^{r}u_s,\quad \mathrm{W}_1=\sum_{r+1}^{3r+1}u_s,\ \text{etc.},$$

and let $$y_x=\sum_{p}^{(2r+1)x-r-1}u_s,$$

where p is some number independent of x. From these definitions we have at once

$$\Delta y_x=\mathrm{W}_x$$

and $$u_0=y_{\frac12+\frac{1}{2(2r+1)}}-y_{\frac12-\frac{1}{2(2r+1)}}.$$

In Bessel's formula,

$$y_{\frac12+m}=\frac{y_0+y_1}{2}+m\Delta y_0+\frac{m^2-\frac14}{2!}\frac{\Delta^2y_{-1}+\Delta^2y_0}{2}+\frac{m(m^2-\frac14)}{3!}\Delta^3y_{-1}+\dots$$

put $$m=\frac{1}{2(2r+1)}.$$

Form the difference $y_{\frac12+m}-y_{\frac12-m}$ and in the result substitute W and its differences for Δy and its differences. We thus obtain the required formula.

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* *J.I.A.* **43**, p. 114.

The result is

$$u_0 = 2m\mathrm{W}_0 + \frac{m(m^2 - \frac{1}{4})}{3!/2}\Delta^2\mathrm{W}_{-1} + \frac{m(m^2 - \frac{1}{4})(m^2 - \frac{9}{4})}{5!/2}\Delta^4\mathrm{W}_{-2} + \dots$$

which, when $2r + 1 = 5$, becomes

$$u_0 = 0{\cdot}2\mathrm{W}_0 - 0{\cdot}008\Delta^2\mathrm{W}_{-1} + 0{\cdot}000896\Delta^4\mathrm{W}_{-2} + \dots$$

as found above.

34. **Inverse Interpolation.**—We shall now consider the process which is the inverse of direct interpolation, namely, that of finding the value of the *argument* corresponding to a given value of the *function* intermediate between two tabulated values, when a difference table of the function is given. This is known as *inverse interpolation.*

Let $f(a + xw)$ denote a particular value of the function of which the differences are tabulated. We now wish to find the value of the argument x corresponding to $f(a + xw)$; for this purpose it is best, if $-\frac{1}{4} < x < \frac{1}{4}$, to use Stirling's formula *

$$\begin{aligned} f(a + xw) = f(a) &+ x\tfrac{1}{2}\{\Delta f(a) + \Delta f(a - w)\} + \tfrac{1}{2}x^2\Delta^2 f(a - w) \\ &+ \tfrac{1}{6}x(x^2 - 1^2)\tfrac{1}{2}\{\Delta^3 f(a - w) + \Delta^3 f(a - 2w)\} \\ &+ \tfrac{1}{24}x^2(x^2 - 1^2)\Delta^4 f(a - 2w) + \dots \qquad (1) \end{aligned}$$

Dividing throughout by $\frac{1}{2}\{\Delta f(a) + \Delta f(a - w)\}$, the coefficient of x, equation (1) may be written in the form

$$x = m - \tfrac{1}{2}x^2\mathrm{D}_1 - \tfrac{1}{6}x(x^2 - 1)\mathrm{D}_2 - \tfrac{1}{24}x^2(x^2 - 1)\mathrm{D}_3 - \dots \qquad (2)$$

where $m = \{f(a + xw) - f(a)\}/\frac{1}{2}\{\Delta f(a) + \Delta f(a - w)\}$,

$\mathrm{D}_1 = \{\Delta^2 f(a - w\}/\frac{1}{2}\{\Delta f(a) + \Delta f(a - w)\}$,

$\mathrm{D}_2 = \{\Delta^3 f(a - w) + \Delta^3 f(a - 2w)\}/\{\Delta f(a) + \Delta f(a - w)\}$,

and so on. We have now to solve equation (2) by successive approximations. 1st approximation: $x = m$. Substituting this value in equation (2) we obtain the 2nd approximation:

$$x = m - \tfrac{1}{2}m^2\mathrm{D}_1 - \tfrac{1}{6}m(m^2 - 1)\mathrm{D}_2 - \tfrac{1}{24}m^2(m^2 - 1)\mathrm{D}_3 - \dots$$

This value of x is now substituted in equation (2) to form the 3rd approximation for x, and so on for further approximations.†

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* If $\frac{1}{4} < x < \frac{3}{4}$, Bessel's formula should be used.

† An excellent method of performing inverse interpolation with a calculating machine is described by L. J. Comrie in the *Nautical Almanac* for 1937, p. 934.

Instead of solving equation (2) by successive approximations we may arrange it in the form

$$m = x + \tfrac{1}{2}x^2 D_1 + \tfrac{1}{6}x(x^2 - 1)D_2 + \tfrac{1}{24}x^2(x^2 - 1)D_3 + \ldots$$

We have merely to reverse this series to obtain a formula from which x may be found by direct substitution, namely,

$$\begin{aligned} x = {} & m(1 + \tfrac{1}{6}D_2 + \ldots) + m^2(-\tfrac{1}{2}D_1 + \tfrac{1}{24}D_3 - \tfrac{1}{4}D_1 D_2 - \ldots) \\ & + m^3(\tfrac{1}{2}D_1{}^2 - \tfrac{1}{6}D_2 - \ldots) \\ & + \ldots \end{aligned}$$

As an example of inverse interpolation, suppose we wish to find the positive root of the equation *

$$z^7 + 28z^4 - 480 = 0.$$

Writing $y = z^7 + 28z^4 - 480$, and finding by a rough graph that the root is a little over 1·9, we construct the following difference table:

z.	y.	Δ.	Δ^2.	Δ^3.
1·90	− 25·7140261			
		11·0886094		
1·91	− 14·6254167		0·2293434	
		11·3179528		0·0041112
1·92	− 3·3074639		0·2334546	
		11·5514074		0·0041775
1·93	8·2439435		0·2376321	
		11·7890395		
1·94	20·0329830			

Evidently the root lies between 1·92 and 1·93, and therefore if the root be $1{\cdot}92 + 0{\cdot}01x$, we have by Stirling's formula in equation (1):

$$\begin{aligned} 0 &= -3{\cdot}3074639 + 11{\cdot}4346801x + 0{\cdot}1167273x^2 + 0{\cdot}0006907(x^3 - x), \\ 0 &= -3{\cdot}3074639 + 11{\cdot}4339894x + 0{\cdot}1167273x^2 + 0{\cdot}0006907x^3. \end{aligned}$$

Dividing throughout by the coefficient of x,

$$x = 0{\cdot}28926595 - 0{\cdot}0102088x^2 - 0{\cdot}0000604x^3.$$

1st approximation : $x = 0{\cdot}28926595$,
2nd approximation : $x = 0{\cdot}28926595 - 0{\cdot}0102088 \times 0{\cdot}083675 - 0{\cdot}0000604 \times 0{\cdot}0242$
$= 0{\cdot}28841027$,
3rd approximation : $x = 0{\cdot}28926595 - 0{\cdot}0102088 \times 0{\cdot}0831805 - 0{\cdot}0000604 \times 0{\cdot}0240$
$= 0{\cdot}28841533$.

The required root is 1·9228841533, correctly to 10 decimal places.

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* This equation was suggested by W. B. Davis (*Ed. Times*, 1867, p. 108) but solved otherwise by him.

35. The Derivatives of a Function.—From the Gregory-Newton formula

$$f(a+xw)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2\,!}\Delta^2 f(a)$$
$$+\frac{x(x-1)\,(x-2)}{3!}\Delta^3 f(a)+\ldots \quad (1)$$

we have at once

$$\frac{f(a+xw)-f(a)}{xw}$$
$$=\frac{1}{w}\{\Delta f(a)+\frac{x-1}{2}\Delta^2 f(a)+\frac{(x-1)\,(x-2)}{3!}\Delta^3 f(a)+\ldots\}.$$

If x is taken very small so that $xw \rightarrow 0$, the left-hand side of the equation is of the form $\{f(a+h)-f(a)\}/h$. The limiting value of this expression when $h \rightarrow 0$ is the *derivative* of the function $f(x)$ for the value a of its argument. We thus obtain

$$f'(a)=\frac{1}{w}\{\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a)-\tfrac{1}{4}\Delta^4 f(a)+\ldots\}. \quad (2)$$

The successive derivatives of the function may be obtained by the use of the differential calculus in the following way. Differentiating (1), we obtain

$$wf'(a+xw)=\Delta f(a)+\frac{2x-1}{2!}\Delta^2 f(a)+\frac{3x^2-6x+2}{3\,!}\Delta^3 f(a)$$
$$+\frac{4x^3-18x^2+22x-6}{4!}\Delta^4 f(a)+\ldots$$

Also

$$w^2 f''(a+xw)=\Delta^2 f(a)+(x-1)\Delta^3 f(a)+\frac{6x^2-18x+11}{12}\Delta^4 f(a)+\ldots$$

and so on for derivatives of higher order.

Putting $x=0$ in this set of equations, we obtain the results

$$wf'(a)=\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a)-\tfrac{1}{4}\Delta^4 f(a)+\tfrac{1}{5}\Delta^5 f(a)$$
$$-\tfrac{1}{6}\Delta^6 f(a)+\ldots$$
$$w^2 f''(a)=\Delta^2 f(a)-\Delta^3 f(a)+\tfrac{11}{12}\Delta^4 f(a)-\tfrac{5}{6}\Delta^5 f(a)+\tfrac{137}{180}\Delta^6 f(a)-\ldots$$
$$w^3 f'''(a)=\Delta^3 f(a)-\tfrac{3}{2}\Delta^4 f(a)+\tfrac{7}{4}\Delta^5 f(a)-\tfrac{15}{8}\Delta^6 f(a)+\ldots$$
$$w^4 f^{\text{iv}}(a)=\Delta^4 f(a)-2\Delta^5 f(a)+\tfrac{17}{6}\Delta^6 f(a)-\ldots$$
$$w^5 f^{\text{v}}(a)=\Delta^5 f(a)-\tfrac{5}{2}\Delta^6 f(a)+\ldots$$
$$w^6 f^{\text{vi}}(a)=\Delta^6 f(a)-\ldots$$

✓*Ex.—To find the first and second derivatives of $\log_e x$ at $x = 500$.*

x.	$\log_e x$.	Δ.	Δ^2.	Δ^3.	Δ.
500	6·214608				
		19803			
510	6·234411		−385		
		19418		15	
520	6·253829		−370		−1
		19048		14	
530	6·272877		−356		−1
		18692		13	
540	6·291569		−343		
		18349			
550	6·309918				

Here $w = 10$ and

$$10f'(500) = 0{\cdot}019803 + \tfrac{1}{2}(0{\cdot}000385) + \tfrac{1}{3}(0{\cdot}000015)$$
$$= 0{\cdot}020001.$$

Also $$100f''(500) = -0{\cdot}000385 - 0{\cdot}000015 - \tfrac{11}{12}(0{\cdot}000001)$$
$$= -0{\cdot}000401.$$

Neglecting the last figure, which is liable to error, we obtain the results

$$f'(500) = 0{\cdot}002000$$
$$f''(500) = -0{\cdot}0000040.$$

We may find the formula for the nth derivative of a function otherwise, by using symbolic operators and expanding the function $f(a+w)$ by Taylor's Theorem.

Thus $$f(a+w) = f(a) + wf'(a) + \frac{w^2}{2!}f''(a) + \ldots \tag{1}$$

If we denote $\dfrac{d}{dx}$, the operator for differentiation, by D, equation (1) becomes

$$f(a+w) = \left(1 + w\mathrm{D} + \frac{w^2\mathrm{D}^2}{2!} + \frac{w^3\mathrm{D}^3}{3!} + \ldots\right)f(a),$$

or $$(1+\Delta)f(a) = e^{w\mathrm{D}}f(a),$$

and $$1 + \Delta \equiv e^{w\mathrm{D}}. \tag{2}$$

Taking logarithms of each side of this equation,

$$w\mathrm{D} = \log_e(1+\Delta)$$
$$= \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots$$

or $$wf'(a) = \Delta f(a) - \tfrac{1}{2}\Delta^2 f(a) + \tfrac{1}{3}\Delta^3 f(a) - \ldots \tag{3}$$

Also $$w^2\mathrm{D}^2 = \{\log(1+\Delta)\}^2$$
$$= (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots)^2.$$

Therefore $$w^2 f''(a) = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots)^2 f(a)$$
$$= \Delta^2 f(a) - \Delta^3 f(a) + \tfrac{11}{12}\Delta^4 f(a) + \ldots \tag{4}$$

and in general

$$w^n f^{(n)}(a) = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \ldots)^n f(a). \tag{5}$$

36. The Derivatives of a Function expressed in Terms of Differences which are in the same Horizontal Line.— By differentiating Stirling's formula,

$$\begin{aligned} f(a+xw) = f(a) &+ x\tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} + \tfrac{1}{2}x^2\Delta^2 f(a-w) \\ &+ \tfrac{1}{6}x(x^2-1^2)\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{24}x^2(x^2-1^2)\Delta^4 f(a-2w) \\ &+ \tfrac{1}{120}x(x^2-1^2)(x^2-2^2)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} \\ &+ \tfrac{1}{720}x^2(x^2-1^2)(x^2-2^2)\Delta^6 f(a-3w), \end{aligned}$$

the differential coefficients may be represented by a rapidly converging series in terms of the horizontal differences. Thus

$$\begin{aligned} &wf'(a+xw) \\ &\quad= \tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} + x\Delta^2 f(a-w) \\ &\quad+ \tfrac{1}{6}(3x^2-1)\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &\quad+ \tfrac{1}{24}(4x^3-2x)\Delta^4 f(a-2w) \\ &\quad+ \tfrac{1}{120}(5x^4-15x^2+4)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \end{aligned}$$

$$\begin{aligned} &w^2 f''(a+xw) \\ &\quad= \Delta^2 f(a-w) + x\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &\quad+ \tfrac{1}{24}(12x^2-2)\Delta^4 f(a-2w) \\ &\quad+ \tfrac{1}{120}(20x^3-30x)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \end{aligned}$$

Putting $x=0$ in these equations, we have

$$\begin{aligned} wf'(a) = \tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} &- \tfrac{1}{6}\cdot\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{30}\cdot\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \end{aligned} \tag{1}$$

$$w^2 f''(a) = \Delta^2 f(a-w) - \tfrac{1}{12}\Delta^4 f(a-2w) + \tfrac{1}{90}\Delta^6 f(a-3w) + \ldots \tag{2}$$

These equations give the value of the derivatives in terms of differences which are symmetrical as regards the direction of increasing and decreasing arguments.

In order to extend these results to derivatives of higher order we shall write Stirling's formula in the central-difference notation of § 20 as far as differences of the eighth order.

$$\begin{aligned} f(a+xw) = u_0 &+ x\mu\delta u_0 + \tfrac{1}{2}x^2\delta^2 u_0 + \tfrac{1}{6}x(x^2-1)\mu\delta^3 u_0 + \tfrac{1}{24}x^2(x^2-1)\delta^4 u_0 \\ &+ \tfrac{1}{120}x(x^2-1)(x^2-4)\mu\delta^5 u_0 + \tfrac{1}{720}x^2(x^2-1)(x^2-4)\delta^6 u_0 \\ &+ \tfrac{1}{5040}x(x^2-1)(x^2-4)(x^2-9)\mu\delta^7 u_0 \\ &+ \tfrac{1}{40320}x^2(x^2-1)(x^2-4)(x^2-9)\delta^8 u_0. \end{aligned}$$

When the right-hand side is arranged according to ascending powers of x, we obtain

$$\begin{aligned} f(a+xw) = u_0 &+ x(\mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0) \\ &+ x^2(\tfrac{1}{2}\delta^2 u_0 - \tfrac{1}{24}\delta^4 u_0 + \tfrac{1}{180}\delta^6 u_0 - \tfrac{1}{1120}\delta^8 u_0) \\ &+ x^3(\tfrac{1}{6}\mu\delta^3 u_0 - \tfrac{1}{24}\mu\delta^5 u_0 + \tfrac{7}{720}\mu\delta^7 u_0) \\ &+ x^4(\tfrac{1}{24}\delta^4 u_0 - \tfrac{1}{144}\delta^6 u_0 + \tfrac{7}{5760}\delta^8 u_0) \\ &+ x^5(\tfrac{1}{120}\delta^5 u_0 - \tfrac{1}{360}\mu\delta^7 u_0) + x^6(\tfrac{1}{720}\delta^6 u_0 - \tfrac{1}{2880}\delta^8 u_0). \end{aligned} \quad (3)$$

If both sides of this equation are differentiated and we substitute the value $x=0$, we obtain the value of $wf'(a)$ as in equation (1); and the higher derivatives of $f(a)$ are formed by differentiating $wf'(a+xw)$, $w^2 f''(a+xw)$, and so on.

The successive derivatives of $f(a)$ correct to differences of the eighth order are given by the following equations:

$$\begin{aligned} wf'(a) &= \mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0, \\ w^2 f''(a) &= \delta^2 u_0 - \tfrac{1}{12}\delta^4 u_0 + \tfrac{1}{90}\delta^6 u_0 - \tfrac{1}{560}\delta^8 u_0, \\ w^3 f'''(a) &= \mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \tfrac{7}{120}\mu\delta^7 u_0, \\ w^4 f^{(\text{iv})}(a) &= \delta^4 u_0 - \tfrac{1}{6}\delta^6 u_0 + \tfrac{7}{240}\delta^8 u_0, \\ w^5 f^{(\text{v})}(a) &= \mu\delta^5 u_0 - \tfrac{1}{3}\mu\delta^7 u_0, \\ w^6 f^{(\text{vi})}(a) &= \delta^6 u_0 - \tfrac{1}{4}\delta^8 u_0. \end{aligned}$$

We see that $wf'(a)$ is equal to the coefficient of x in (3) and, in general, *$w^n f^{(n)}(a)$ is equal to the coefficient of x^n in the equation* (3) *multiplied by* $n!$. This result might have been obtained at once by comparing (3) with Taylor's expansion of $f(a+xw)$.

37. To express the Derivatives of a Function in Terms of its Divided Differences.—We shall first find the derivative of a function $f(x)$ for the particular value a_0 of the argument x in terms of its divided differences. As shown at equation (3), § 13, we may write

$$\begin{aligned} f(u,a_0) = f(a_0\, a_1) &+ (u-a_1)f(a_0,a_1,a_2) + (u-a_1)(u-a_2)f(a_0,a_1,a_2\, a_3) \\ &+ \ldots + (u-a_1)(u-a_2)\ \ldots\ (u-a_{n-1})f(a_0, a_1, \ldots, a_n), \end{aligned}$$

where the divided differences of order beyond the nth are supposed negligible. If we put $u=a_0$, we have

$$\begin{aligned} f(a_0, a_0) = f(a_0, a_1) &+ (a_0-a_1)f(a_0, a_1, a_2) \\ &+ (a_0-a_1)(a_0-a_2)f(a_0, a_1, a_2, a_3) + \ldots \\ &+ (a_0-a_1)(a_0-a_2)\ \ldots\ (a_0-a_{n-1})f(a_0, a_1, \ldots, a_n). \end{aligned} \quad (1)$$

But in § 16 we found that

$$f(u) = f(a_0) + (u - a_0)f(a_0, a_0) + (u - a_0)^2 f(a_0, a_0, a_0) \\ + (u - a_0)^3 f(a_0, a_0, a_0, a_0) + \dots$$

and by Taylor's expansion,

$$f(u) = f(a_0) + (u - a_0)f'(a_0) + (u - a_0)^2 \frac{f''(a_0)}{2\,!} + (u - a_0)^3 \frac{f'''(a_0)}{3\,!} + \dots$$

so that $f'(a_0) = f(a_0\ a_0)$, $\tfrac{1}{2}f''(a_0) = f(a_0, a_0, a_0)$, and in general

$$f^{(n)}(a_0)/n\,! = f(a_0, a_0, \dots, a_0),$$

which gives the nth derivative in terms of the divided difference of the nth order with repeated arguments.

Equation (1) thus becomes

$$f'(a_0) = f(a_0\ a_1) + (a_0 - a_1)f(a_0, a_1, a_2) + (a_0 - a_1)(a_0 - a_2)f(a_0, a_1, a_2, a_3) \\ + \dots + (a_0 - a_1)(a_0 - a_2) \dots (a_0 - a_{n-1})f(a_0, a_1, \dots, a_n), \quad (2)$$

which gives $f'(a_0)$ in terms of its successive divided differences.

As a special case of this formula when $a_1 = a_0 + w$, $a_2 = a_0 + 2w$, etc.

$$f'(a_0) = \frac{1}{w}\Delta f(a_0) + (-w)\frac{1}{2w^2}\Delta^2 f(a_0) + (-w)(-2w)\frac{1}{3\,!w^3}\Delta^3 f(a_0) + \dots$$

or

$$wf'(a_0) = \Delta f(a_0) - \tfrac{1}{2}\Delta^2 f(a_0) + \tfrac{1}{3}\Delta^3 f(a_0) - \dots$$

which is the formula of § 35.

A more general expression for the derivatives of a function in terms of its divided differences may be obtained from Newton's formula:

$$f(x) = f(a_0) + (x - a_0)f(a_0, a_1) + (x - a_0)(x - a_1)f(a_0, a_1, a_2) \\ + (x - a_0)(x - a_1)(x - a_2)f(a_0, a_1, a_2, a_3) + \dots$$

Denoting the factor $(x - a_n)$ by α_n, this equation becomes

$$f(x) = f(a_0) + \alpha_0 f(a_0, a_1) + \alpha_0\alpha_1 f(a_0, a_1, a_2) + \alpha_0\alpha_1\alpha_2 f(a_0, a_1, a_2, a_3) \\ + \dots + \alpha_0\alpha_1\alpha_2 \dots \alpha_{n-1} f(a_0, a_1, a_2, \dots, a_n). \quad (3)$$

Differentiating both sides of this equation, we see that

$$f'(x) = f(a_0, a_1) + (\alpha_0 + \alpha_1)f(a_0, a_1, a_2) \\ + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2)f(a_0, a_1, a_2, a_3) + \dots \quad (4)$$

$$f''(x)/2\,! = f(a_0, a_1, a_2) + (\alpha_0 + \alpha_1 + \alpha_2)f(a_0, a_1, a_2, a_3) \\ + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_0\alpha_3 + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)f(a_0, a_1, a_2, a_3, a_4) + \dots$$

$$f'''(x)/3\,! = f(a_0, a_1, a_2, a_3) \\ + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)f(a_0, a_1, a_2, a_3, a_4) + \dots \quad (5)$$

$$f^{\text{iv}}(x)/4\,! = f(a_0, a_1, a_2, a_3, a_4) \\ + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)f(a_0, a_1, a_2, a_3, a_4, a_5) + \dots \quad (6)$$

and so on.

In these equations the coefficient of the divided differences of order r is a symmetric function of the quantities $a_0, a_1, a_2, \ldots, a_{r-1}$. In equation (3) this coefficient is of r dimensions, and after each differentiation its dimensions decrease by unity; so we see, therefore, that the coefficient of $f(a_0, a_1, \ldots, a_r)$ in the equation for $f^{(r)}(x)/r!$ is unity (*i.e.* zero dimension in $a_0, a_1, \ldots, a_{r-1}$), and all differences of lower order vanish.

If we suppose $a_0 = a_1 = a_2 = a_3 = \ldots = a_n$, we obtain the values given above: $f'(a_0) = f(a_0, a_0)$, $f''(a_0) = 2f(a_0, a_0, a_0)$, and so on.

Substituting in equation (4) the value $x = a_0$, we obtain equation (2), namely,

$$f'(a_0) = f(a_0, a_1) + (a_0 - a_1)f(a_0, a_1, a_2) + (a_0 - a_1)(a_0 - a_2)f(a_0, a_1, a_2, a_3) + \ldots$$

The latter equation is used when the derivative of a single value of the function is required; but when the derivatives of several values of the function are to be computed, we use equation (4).

Ex.—From the following table of values compute the third and fourth derivatives of $f(\theta)$ when the argument θ has the values 5, 14, *and* 23 *respectively.*

θ	2	4	9	13	16	21	29
$f(\theta)$	57	1345	66340	402052	1118209	4287844	21242820

We first form a table of divided differences:

θ.	$f(\theta)$.					
$a_0 = 2$	57					
		644				
$a_1 = 4$	1345		1765			
		12999		556		
$a_2 = 9$	66340		7881		45	
		83928		1186		1
$a_3 = 13$	402052		22113		64	
		238719		2274		1
$a_4 = 16$	1118209		49401		89	
		633927		4054		
$a_5 = 21$	4287844		114265			
		2119372				
$a_6 = 29$	21242820					

The function is evidently a polynomial of the 5th degree. Tabulating the values of $a_0, a_1, a_2, \ldots$, we find

	$\theta = 5$.	$\theta = 14$.	$\theta = 23$.
a_0	3	12	21
a_1	1	10	19
a_2	−4	5	14
a_3	−8	1	10
a_4	−11	−2	7

From equation (5) we have at once

$$\tfrac{1}{6}f'''(\theta) = 556 + (a_0 + a_1 + a_2 + a_3)45$$
$$+ (a_0a_1 + a_0a_2 + a_0a_3 + a_0a_4 + a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)1,$$

so $$f'''(5) = 1\,626, \quad f'''(14) = 12\,102, \quad f'''(23) = 32\,298.$$

From equation (6) we have

$$\tfrac{1}{24}f^{\text{IV}}(\theta) = 45 + (a_0 + a_1 + a_2 + a_3 + a_4)1,$$

so $$f^{\text{IV}}(5) = 624, \quad f^{\text{IV}}(14) = 1704, \quad f^{\text{IV}}(23) = 2784.$$

EXAMPLES ON CHAPTER IV

1. The logs of the numbers 400, 410, 420, 430, 440 being given to seven places of decimals, find the logs of the integers between 400 and 410.

$\log 400 = 2{\cdot}6020600$
$\log 410 = 2{\cdot}6127839$
$\log 420 = 2{\cdot}6232493$
$\log 430 = 2{\cdot}6334685$
$\log 440 = 2{\cdot}6434527$

2. If $\Delta^r T_0$ is the last non-zero difference of the original sequence, so that $\Delta^{r+1}T_0 = 0$, $\Delta^{r+2}T_0 = 0$, . . ., show that the formulae for sub-tabulation are:

$$\Delta_1{}^r t_0 = \frac{1}{m^r}\Delta^r T_0,\text{*}$$

$$\Delta_1{}^{r-1} t_0 = \frac{1}{m^{r-1}}\Delta^{r-1} T_0 + \frac{(r-1)(1-m)}{2m^r}\Delta^r T_0,$$

$$\Delta_1{}^{r-2} t_0 = \frac{1}{m^{r-2}}\Delta^{r-2} T_0 + \frac{(r-2)(1-m)}{2m^{r-1}}\Delta^{r-1} T_0$$
$$+ \left\{\frac{(r-2)(1-m)(1-2m)}{2.3.m^r} + \frac{(r-2)(r-3)(1-m)^2}{8m^r}\right\}\Delta^r T_0.$$

The differences of order higher than the rth in the new sequence are, of course, all zero.

3. The following are three consecutive quinquennial sums:

44133, 41921 and 39387.

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* Mouton, an astronomer of Lyons, in 1670 noticed that if in a sequence whose rth differences are constant, say $= c$, intermediate terms are inserted corresponding to a division of each interval of the argument into m equal parts, then the new sequence has its rth difference constant and equal to c/m^r.

Find the value of the quantity for the middle year of the second quinquennium.

4. The populations for four consecutive age groups are given by the table of values

Age Group.	*Population.*
25 to 29 years (inclusive)	458572
30 to 34 years („)	441424
35 to 39 years („)	423123
40 to 44 years („)	402918

Estimate the populations of ages between 32 and 33 years, and between 37 and 38 years respectively.

5. Show that if

$$W_0 = u_{0/t} + u_{1/t} + \cdots + u_{(t-1)/t},$$

and in general

$$W_x = u_{x+\frac{0}{t}} + u_{x+\frac{1}{t}} + \cdots + u_{x+\frac{(t-1)}{t}},$$

then the individual value $u_{x/t}$ may be found from the groups of t individual values $W_0, W_1, W_2, \ldots$ and their differences by the formula

$$u_{x/t} = \frac{W_0}{t} + (2x - t + 1)\frac{\Delta W_0}{t^2.2\,!} + \left\{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\right\}\frac{\Delta^2 W_0}{t^3.3\,!},$$

where third differences are neglected.*

✓ 6. In the following set of data h is the height above sea-level, p the barometric pressure. Calculate by a difference table the height at which $p = 29$ and the pressure when $h = 5280$.

$h=$	0	2753	4763	6942	10593
$p=$	30	27	25	23	20

✓ 7. Form a difference table from the following steam data, where p is pressure in lbs. per square inch.

$\theta°$ C	93·0	96·2	100·0	104·2	108·7
p	11·38	12·80	14·70	17·07	19·91

Calculate p when $\theta = 99°{\cdot}1$ and determine by inverse interpolation the temperature at which $p = 15$.

8. Calculate the real root of the equation

$$x^3 + x - 3 = 0$$

by inverse interpolation.

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* C. H. Forsyth, *Quarterly Publications of the American Statistical Association*, December 1916.

9. Find the differential coefficient of $\log_e x$ at $x=300$, given the table of values

x.	$\log_e x$.
300	5·703782474656
301	5·707110264749
302	5·710427017375
303	5·713732805509
304	5·717027701406
305	5·720311776607
306	5·723585101952
307	5·726847747587

Find from the above table the differential coefficient of $\log_e x$ at $x=302$.

10. Given the values

x.	y.
0	858·313740 095
1	869·645772 308
2	880·975826 766
3	892·303904 583
4	903·630006 875

find the value of $\dfrac{d^2y}{dx^2}$ when $x=0$.

✓11. Find $\dfrac{d^2y}{dz^2}$ when $z=1$, given the following values:

z.	y.
1	0·198669
2	0·295520
3	0·389418
4	0·479425
5	0·564642
6	0·644217

12. Apply the central-difference formulae of § 36 to compute the first and second derivatives of $\log_e 304$, having given the table of values of Ex. 9.

✓13. From the following data compute the first four derivatives of the function y corresponding to the argument $x=11$:

x.	y.
2	108 243 219
5	121 550 628
9	141 158 164
13	163 047 364
15	174 900 628
21	214 358 884

CHAPTER V

DETERMINANTS AND LINEAR EQUATIONS

38. The Numerical Computation of Determinants.—In this chapter we shall consider the problem of finding the numerical value of a determinant, say,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} \qquad \text{(A)}$$

when the elements a_1, a_2, . . . are given numbers. The method generally adopted, which is due to Chiò,* is as follows:

We first notice whether any element is equal to unity; if not, we prepare the determinant for our subsequent operations by multiplying some row or column by such a number p as will make one of the elements unity, and put $1/p$ as a factor outside the determinant. This unit element will henceforth be called the *pivotal element.* Thus in the above determinant we shall suppose that $b_3 = 1$ and take b_3 as the pivotal element. *We shall show that the above determinant is equal to the determinant*

$$(-1)^{2+3} \begin{vmatrix} a_1 - a_3b_1 & a_2 - a_3b_2 & a_4 - a_3b_4 & a_5 - a_3b_5 \\ c_1 - c_3b_1 & c_2 - c_3b_2 & c_4 - c_3b_4 & c_5 - c_3b_5 \\ d_1 - d_3b_1 & d_2 - d_3b_2 & d_4 - d_3b_4 & d_5 - d_3b_5 \\ e_1 - e_3b_1 & e_2 - e_3b_2 & e_4 - e_3b_4 & e_5 - e_3b_5 \end{vmatrix} \qquad \text{(B)}$$

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* F. Chiò, *Mémoire sur les fonctions connues sous le nom de résultantes ou de déterminants*, Turin (1853).

The law of formation of this new determinant (B) may be expressed thus: *The row and column intersecting in the pivotal element of the original determinant, say the* r*th row and* s*th column, are deleted; then every element* y *is diminished by the product of the elements which stand where the eliminated row and column are met by perpendiculars from* y, *and the whole determinant is multiplied by* $(-1)^{r+s}$.

The advantage gained by substituting the determinant (B) for the determinant (A) is that *(B) is of order one unit lower than A*; and therefore by repeated application of this method of reduction *we can reduce any determinant to the second order*, when its value may be written down at once.

To prove this theorem, we first divide the columns of the determinant (A) by $b_1, b_2, \ldots, b_5$ respectively, so that it takes the form

$$b_1 b_2 b_4 b_5 \begin{vmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_2} & a_3 & \frac{a_4}{b_4} & \frac{a_5}{b_5} \\ 1 & 1 & 1 & 1 & 1 \\ \frac{c_1}{b_1} & \frac{c_2}{b_2} & c_3 & \frac{c_4}{b_4} & \frac{c_5}{b_5} \\ \frac{d_1}{b_1} & \frac{d_2}{b_2} & d_3 & \frac{d_4}{b_4} & \frac{d_5}{b_5} \\ \frac{e_1}{b_1} & \frac{e_2}{b_2} & e_3 & \frac{e_4}{b_4} & \frac{e_5}{b_5} \end{vmatrix}$$

since $b_3 = 1$: then (subtracting the elements of the third column from those in the other columns) we write the determinant in the form

$$b_1 b_2 b_4 b_5 \begin{vmatrix} \frac{a_1}{b_1} - a_3 & \frac{a_2}{b_2} - a_3 & a_3 & \frac{a_4}{b_4} - a_3 & \frac{a_5}{b_5} - a_3 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{c_1}{b_1} - c_3 & \frac{c_2}{b_2} - c_3 & c_3 & \frac{c_4}{b_4} - c_3 & \frac{c_5}{b_5} - c_3 \\ \frac{d_1}{b_1} - d_3 & \frac{d_2}{b_2} - d_3 & d_3 & \frac{d_4}{b_4} - d_3 & \frac{d_5}{b_5} - d_3 \\ \frac{e_1}{b_1} - e_3 & \frac{e_2}{b_2} - e_3 & e_3 & \frac{e_4}{b_4} - e_3 & \frac{e_5}{b_5} - e_3 \end{vmatrix}$$

This determinant may now be written

$$(-1)^{2+3} b_1 b_2 b_4 b_5 \begin{vmatrix} \frac{a_1}{b_1} - a_3 & \frac{a_2}{b_2} - a_3 & \frac{a_4}{b_4} - a_3 & \frac{a_5}{b_5} - a_3 \\ \frac{c_1}{b_1} - c_3 & \frac{c_2}{b_2} - c_3 & \frac{c_4}{b_4} - c_3 & \frac{c_5}{b_5} - c_3 \\ \frac{d_1}{b_1} - d_3 & \frac{d_2}{b_2} - d_3 & \frac{d_4}{b_4} - d_3 & \frac{d_5}{b_5} - d_3 \\ \frac{e_1}{b_1} - e_3 & \frac{e_2}{b_2} - e_3 & \frac{e_4}{b_4} - e_3 & \frac{e_5}{b_5} - e_3 \end{vmatrix}$$

which is obviously equivalent to the form (B).

It is usually advisable to prepare the determinant for computation by forming zero elements in the row and column containing the pivotal element. For example, in the determinant (A) of Ex. 1 below, zero elements may be introduced into the first row by adding three times the third column to the first column to form the new first column; then adding the third column to the second column to form the new second column, and so on. In this way the subsequent calculations are simplified.

In performing the computation of a determinant it will be found convenient to draw pencil lines through the row and column which intersect in the pivotal element; this helps the eye in finding the elements at the feet of the perpendiculars.

Instead of first dividing some row or column in order to obtain a pivotal element equal to unity, we may eliminate the row and column intersecting in *any* element (not necessarily unity) as follows: Delete the row and column in question by drawing pencil lines through them; then the rule is

$$\textit{New element} = \textit{old element} - \frac{\textit{product of elements at feet of perpendiculars}}{\textit{element at intersection of pencil lines}},$$

the new determinant being multiplied by the element at the intersection of the pencil lines and by the factor $(-1)^{r+s}$, *where* r *and* s *denote the numbers of the deleted row and column respectively.*

The above method of computing the value of a determinant enables us at the same time to compute the co-factors of the elements corresponding to the surviving elements; for these co-factors are actually equal to the co-factors of the corresponding elements in the reduced determinant.

For example, the co-factor of c_1 in determinant (A) is

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ d_2 & d_3 & d_4 & d_5 \\ e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

which, since $b_3 = 1$, is equal to

$$\begin{vmatrix} a_2 - a_3b_2 & a_4 - a_3b_4 & a_5 - a_3b_5 \\ d_2 - d_3b_2 & d_4 - d_3b_4 & d_5 - d_3b_5 \\ e_2 - e_3b_2 & e_4 - e_3b_4 & e_5 - e_3b_5 \end{vmatrix}$$

the latter determinant being the co-factor of $(c_1 - c_3b_1)$, which is the element in the reduced determinant (B) corresponding to c_1 in (A).

Ex. 1.—*Evaluate*

$$\begin{vmatrix} 3 & 1 & -1 & 2 & 1 \\ -2 & 3 & 1 & 4 & 3 \\ 1 & 4 & 2 & 3 & 1 \\ 5 & -2 & -3 & 5 & -1 \\ -1 & 1 & 2 & 3 & 2 \end{vmatrix} \qquad \text{(A)}$$

We may select as pivotal element the number 1 at the intersection of the first row and the fifth column; the rule then gives

$$\begin{vmatrix} 3 & 1 & -1 & 2 & 1 \\ -2 & 3 & 1 & 4 & 3 \\ 1 & 4 & 2 & 3 & 1 \\ 5 & -2 & -3 & 5 & -1 \\ -1 & 1 & 2 & 3 & 2 \end{vmatrix} = (-1)^6 \begin{vmatrix} -11 & 0 & 4 & -2 \\ -2 & 3 & 3 & 1 \\ 8 & -1 & -4 & 7 \\ -7 & -1 & 4 & -1 \end{vmatrix} \qquad \text{(B)}$$

Taking as new pivotal element the unit at the intersection of the second row and last column, we obtain

$$(-1)^{6+6} \begin{vmatrix} -15 & 6 & 10 \\ 22 & -22 & -25 \\ -9 & 2 & 7 \end{vmatrix} \qquad \text{(C)}$$

As there is now no unit element in this determinant, we may divide the second column by 2 and so form a unit pivotal element. The determinant becomes

$$\begin{aligned} 2 &\begin{vmatrix} -15 & 3 & 10 \\ 22 & -11 & -25 \\ -9 & 1 & 7 \end{vmatrix} \\ &= 2\,(-1)^5 \begin{vmatrix} 12 & -11 \\ -77 & 52 \end{vmatrix} \qquad \text{(D)} \\ &= 446, \text{ the required value.} \end{aligned}$$

If it were required to determine the co-factor of (say) the element in the fourth row and first column of the above determinant, we should have the co-factor of 5 in (A) equal to the co-factor of 8, the corresponding element in the reduced determinant (B)
= the co-factor of 22 in the next reduced determinant (C)
= the co-factor of -77 in (D)
$= -22$.

Ex. 2.—*Evaluate the determinant for the Legendre polynomial of order five:*

$$P_5(z)=\frac{1}{5!}\begin{vmatrix} z & 1 & 0 & 0 & 0 \\ 1 & 3z & 2 & 0 & 0 \\ 0 & 2 & 5z & 3 & 0 \\ 0 & 0 & 3 & 7z & 4 \\ 0 & 0 & 0 & 4 & 9z \end{vmatrix}$$

We eliminate the fourth row and fifth column, noting that the element at the intersection of these lines becomes a factor of the determinant. Thus

$$5!\,P_5(z)=-4\begin{vmatrix} z & 1 & 0 & 0 \\ 1 & 3z & 2 & 0 \\ 0 & 2 & 5z & 3 \\ 0 & 0 & -\frac{27z}{4} & 4-\frac{63z^2}{4} \end{vmatrix}$$

$$\text{(eliminating the 1st row and 2nd col.)}=\ 4\begin{vmatrix} 1-3z^2 & 2 & 0 \\ -2z & 5z & 3 \\ 0 & -\frac{27z}{4} & 4-\frac{63z^2}{4} \end{vmatrix}$$

$$\text{(eliminating the 1st row and 2nd col.)}=-8\begin{vmatrix} \frac{15}{2}z^3-\frac{9z}{2} & 3 \\ -\frac{81z^3}{8}+\frac{27z}{8} & 4-\frac{63z^2}{4} \end{vmatrix}$$

$$=945z^5-1050z^3+225z.$$

So we have $P_5(z)=\frac{1}{8}(63z^5-70z^3+15z)$.

39. **The Solution of a System of Linear Equations.**—Being now in a position to compute the numerical value of a determinant, we can solve a set of linear equations in any number of unknowns, $x_1, x_2, x_3, \ldots, x_n$, say

$$\left\{\begin{array}{l} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=c_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=c_n \end{array}\right\}$$

by the formulae which are proved in works on determinants, namely:

$$x_1=\frac{\begin{vmatrix} c_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ c_2 & a_{22} & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}},\quad x_2=\frac{\begin{vmatrix} a_{11} & c_1 & a_{13} & \ldots & a_{1n} \\ a_{21} & c_2 & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & c_n & a_{n3} & \ldots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}}$$

and similar expressions for $x_3, x_4, \ldots, x_n$.*

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* Further remarks and examples on the solution of linear equations will be found in Chapter IX. in connection with the solution of the "normal equations" in the Method of Least Squares.

Ex.—Find the values of x, y, z, w that satisfy the system of equations:

$$\left.\begin{aligned} 9x+3y+4z+2w&=28 \\ 3x+5z-4w&=12 \\ y+z+w&=5 \\ 6x-y+3w&=19 \end{aligned}\right\}.$$

We have at once

$$x=\frac{\begin{vmatrix} 28 & 3 & 4 & 2 \\ 12 & 0 & 5 & -4 \\ 5 & 1 & 1 & 1 \\ 19 & -1 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 9 & 3 & 4 & 2 \\ 3 & 0 & 5 & -4 \\ 0 & 1 & 1 & 1 \\ 6 & -1 & 0 & 3 \end{vmatrix}}=\frac{-276}{-207}=1\tfrac{1}{3},$$

$$y=-\frac{1}{207}\begin{vmatrix} 9 & 28 & 4 & 2 \\ 3 & 12 & 5 & -4 \\ 0 & 5 & 1 & 1 \\ 6 & 19 & 0 & 3 \end{vmatrix}=-\frac{414}{207}=-2,$$

and in the same way we obtain the values $z=4$, $w=3$.

EXAMPLES ON CHAPTER V

Evaluate the determinants

$$(\alpha)\begin{vmatrix} 2 & 4 & 3 & 7 & 0 \\ 2 & 3 & 5 & 5 & 3 \\ 1 & 0 & 2 & 1 & 2 \\ 3 & 0 & 1 & 0 & 6 \\ 2 & 1 & 2 & 4 & 1 \end{vmatrix} \qquad (\beta)\begin{vmatrix} 1 & 3 & 0 & 0 & 0 \\ -1 & 2 & 3 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

$$(\gamma)\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & \frac{1}{2} & 1 & 0 & 0 \\ 4 & \frac{1}{3} & 2 & 1 & 0 \\ 5 & \frac{1}{4} & 3 & \frac{1}{2} & 1 \\ 6 & \frac{1}{5} & 4 & \frac{1}{3} & 2 \end{vmatrix}$$

2. Show that the determinant

$$\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 & . \\ 2^2 & 5 & 1 & 0 & 0 & 0 & . \\ 0 & 3^2 & 7 & 1 & 0 & 0 & . \\ 0 & 0 & 4^2 & 9 & 1 & 0 & . \\ 0 & 0 & 0 & 5^2 & 11 & 1 & . \\ . & . & . & . & . & . & . \end{vmatrix}$$

with $(n-1)$ rows, is equal to $n!$.

3. Compute the values of

$$(\alpha)\ \begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix} \qquad (\beta)\ \begin{vmatrix} 3 & 7 & 1 & 2 & 5 \\ 6 & 4 & 3 & 0 & 2 \\ 0 & 3 & 0 & 1 & 2 \\ 1 & 0 & 6 & 5 & 3 \\ 2 & 1 & 0 & 2 & 0 \end{vmatrix}$$

$$(\gamma)\ \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{vmatrix}$$

4. Verify the relation

$$2^n = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & . \\ -1 & 1 & 1 & 0 & 0 & . \\ 1 & 1 & 2 & 1 & 0 & . \\ -1 & 1 & 3 & 3 & 1 & . \\ 1 & 1 & 4 & 6 & 4 & . \\ . & . & . & . & . & . \end{vmatrix} (n+1) \text{ rows},$$

the constituents being binomial coefficients, when $n = 2, 3, 4, \ldots$

5. Solve determinantally the following sets of equations:

$$(\alpha) \qquad \left.\begin{aligned} x + 2y - 2z &= 3q \\ 3x + y + 2w &= 4p \\ y + z - w &= 6q \\ x - z - w &= 0 \end{aligned}\right\}.$$

$$(\beta) \qquad \left.\begin{aligned} x_1 + 5x_3 - 2x_4 + x_5 &= 10 \\ 4x_2 - 3x_3 + 7x_4 &= -14 \\ -9x_1 + 8x_2 + 10x_4 &= 31 \\ 2x_1 + 20x_2 - 13x_3 + x_5 &= -2 \\ 5x_1 + 4x_2 + 3x_3 + 2x_4 + 4x_5 &= 1 \end{aligned}\right\}.$$

CHAPTER VI

THE NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

40. Introduction.—In the present chapter we shall show how to find the value of an unknown quantity which satisfies some given algebraical or transcendental equation; or the values of several unknown quantities which satisfy a set of given equations, equal in number to the number of the unknowns.

The methods in use may be classified as follows:

(*a*) *Literal methods,* in which the solution is obtained as a general formula, so that nothing remains but to substitute numerical values in the formula; as, for example, the solution of the quadratic equation $x^2+2bx+c=0$ by the formula

$$x=-b\pm\sqrt{b^2-c}.$$

These literal solutions are valuable when they can be obtained,[*] but in most of the cases we shall have to discuss, they are unattainable, at any rate in a form involving only a finite number of arithmetical operations.

(*β*) *Numerical or computer's methods,* in which the working is mainly arithmetical from the beginning. These are, on the whole, the most useful, particularly when a high degree of accuracy is required, and they constitute the main topic of the present chapter.

(*γ*) *Graphical methods,* in which the solution is obtained by drawing diagrams. These are much used when a rough solution is all that is required or as a preparation for a more accurate solution by numerical methods, but for many purposes they have been superseded by the

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[*] The solution of the cubic discussed in § 62 below, and the general formula proved in § 60, are examples of literal solutions.

(δ) *Nomographic methods*, in which a diagram is prepared once for all to serve for a wide class of cases, so that it may be used over and over again with different numerical data.

(ϵ) *Mechanical methods*, in which some mechanical arrangement is applied; many ingenious machines have been devised for the purpose of solving different equations, but on account of their cost and complexity they have not come into extensive use.

41. **The Pre-Newtonian Period.**—A method for the extraction of the square and cube roots of numbers, digit by digit, was discovered by the Hindu mathematicians, and by them communicated to the Arabs, who transmitted it to Europe.

This method was extended by Vieta in 1600[*] so as to furnish the roots of algebraic equations in general. The process was so laborious that a seventeenth-century mathematician described it as "work unfit for a Christian,"[†] but it was in general use from 1600 to 1680.

In 1674 a method depending on a new principle, the principle of *iteration*, was communicated in a letter from Gregory to Collins;[‡] and independently, a few months later, in a letter from Michael Dary to Newton.[§] This principle we shall now discuss.

42. **The Principle of Iteration.**—As a first illustration of the principle of iteration we shall consider an algorithm suggested by Newton[‖] for the determination of square roots, which may be described as follows:

Let N *be the number whose square root is required. Take any number* x_0 *and from it form* x_1 *according to the equation* $x_1 = \frac{1}{2}(x_0 + N/x_0)$. *From* x_1 *form* x_2 *according to the equation* $x_2 = \frac{1}{2}(x_1 + N/x_1)$. *From* x_2 *form* x_3 *according to the equation*

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[*] *De numerosâ potestatum adfectarum resolutione*, 1600.

[†] Warner in Rigaud's *Correspondence of Scientific Men of the 17th Century*, 1, p. 248.

[‡] Rigaud's *Correspondence*, 2, p. 255.

[§] Rigaud, *op. cit.* 2, p. 365. For an account of Dary see Rigaud, 1, p. 204.

[‖] *Op. cit.* 2, p. 372.

$x_3=\frac{1}{2}(x_2+\mathrm{N}/x_2)$, *and so on. Then the sequence of numbers* $x_0, x_1, x_2, x_3, \ldots$, *tends to a limit which is* $\sqrt{\mathrm{N}}$.

Thus, taking $\mathrm{N}=10$ and $x_0=1$, we have

$$x_1=\tfrac{1}{2}(1+10)=5{\cdot}5,$$
$$x_2=\tfrac{1}{2}(5{\cdot}5+10/5{\cdot}5)=\tfrac{1}{2}(5{\cdot}5+1{\cdot}8)=3{\cdot}7,$$
$$x_3=\tfrac{1}{2}(3{\cdot}7+10/3{\cdot}7)=\tfrac{1}{2}(3{\cdot}70+2{\cdot}7)=3{\cdot}2,$$
$$x_4=\tfrac{1}{2}(3{\cdot}2+10/3{\cdot}2)=\tfrac{1}{2}(3{\cdot}2+3{\cdot}125)=3{\cdot}163,$$
$$x_5=\tfrac{1}{2}(3{\cdot}163+10/3{\cdot}163)=\tfrac{1}{2}(3{\cdot}163+3{\cdot}161555)=3{\cdot}1622775,$$
$$x_6=\tfrac{1}{2}(3{\cdot}1622775+10/3{\cdot}1622775)=\tfrac{1}{2}(3{\cdot}1622775+3{\cdot}1622778)$$
$$=3{\cdot}1622777,$$

which is the square root of 10, correctly to seven decimal places.

In order to prove the validity of the process, we proceed as follows:

The equation $$x_p=\tfrac{1}{2}(x_{p-1}+\mathrm{N}/x_{p-1})$$
may be written
$$\frac{x_p-\sqrt{\mathrm{N}}}{x_p+\sqrt{\mathrm{N}}}=\left(\frac{x_{p-1}-\sqrt{\mathrm{N}}}{x_{p-1}+\sqrt{\mathrm{N}}}\right)^2,$$
whence we have
$$\frac{x_n-\sqrt{\mathrm{N}}}{x_n+\sqrt{\mathrm{N}}}=\left(\frac{x_0-\sqrt{\mathrm{N}}}{x_0+\sqrt{\mathrm{N}}}\right)^{(2^n)}$$

From this equation it is evident that if*
$$\left|\frac{x_0-\sqrt{\mathrm{N}}}{x_0+\sqrt{\mathrm{N}}}\right|<1, \text{ then } \mathrm{Lt}_{n\to\infty}x_n=\sqrt{\mathrm{N}}; \tag{1}$$
if
$$\left|\frac{x_0-\sqrt{\mathrm{N}}}{x_0+\sqrt{\mathrm{N}}}\right|>1, \text{ then } \mathrm{Lt}_{n\to\infty}x_n=-\sqrt{\mathrm{N}}. \tag{2}$$

The limiting case is when
$$|x_0-\sqrt{\mathrm{N}}|=|x_0+\sqrt{\mathrm{N}}|.$$

If we write $$x_0=re^{i\theta},\quad \mathrm{N}=\mathrm{X}e^{i\alpha},$$
this becomes
$$|r\cos\theta-\mathrm{X}^{\frac{1}{2}}\cos\tfrac{1}{2}\alpha+ir\sin\theta-i\mathrm{X}^{\frac{1}{2}}\sin\tfrac{1}{2}\alpha|$$
$$=|r\cos\theta+\mathrm{X}^{\frac{1}{2}}\cos\tfrac{1}{2}\alpha+ir\sin\theta+i\mathrm{X}^{\frac{1}{2}}\sin\tfrac{1}{2}\alpha|,$$
or, squaring both sides,
$$r^2+\mathrm{X}-2r\mathrm{X}^{\frac{1}{2}}\cos(\theta-\tfrac{1}{2}\alpha)=r^2+\mathrm{X}+2r\mathrm{X}^{\frac{1}{2}}\cos(\theta-\tfrac{1}{2}\alpha),$$
so $$\cos(\theta-\tfrac{1}{2}\alpha)=0,$$
or $$\theta=\pm 90^\circ+\tfrac{1}{2}\alpha,$$
which is the equation of a straight line through the origin in

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* If z is a complex number, say equal to $u+v\sqrt{-1}$, where u and v are real numbers, then $\sqrt{(u^2+v^2)}$ is denoted by $|z|$ and is called the *modulus* of z.

the plane of the complex variable x_0, perpendicular to the line joining the points $\sqrt{N}$ and $-\sqrt{N}$. Denoting this line by l, we regard it as dividing the plane into two half-planes; and we see from (1) and (2) above that *the algorithm leads to the value* $\sqrt{N}$, *as long as the initial number* x_0 *is taken in that half of the plane which contains* $\sqrt{N}$; *and the algorithm leads to the value* $-\sqrt{N}$, *as long as* x_0 *is taken in that half-plane which contains* $-\sqrt{N}$. *If* x_0 *is taken exactly on the line* l, *the sequence* $x_0, x_1, x_2, \ldots$, *does not tend to a limit.*[*]

A pleasing characteristic of iterative processes may be observed in connection with this example, namely, that a mistake in the performance of the numerical work does not invalidate the whole calculation. If, for example, a mistake were made in calculating x_1 from x_0, the erroneous value x_1' obtained might have been obtained correctly by starting from a different value x_0'; and since x_0 is to be taken arbitrarily, the true solution may be reached by way of x_1' as well as by way of x_1. The correct result is obtained whenever the numbers $x_r, x_{r+1}, x_{r+2}, \ldots$, are obviously tending to a limit, however many errors may have been committed in obtaining these numbers. This valuable feature of iterative methods has made them very popular.

43. **Geometrical Interpretation of Iteration.**—The nature of iterative methods may readily be illustrated[†] geometrically. Let $f(x)=0$ be the equation. Write it in the form $f_1(x)=f_2(x)$, as may usually be done in many ways; thus, if the equation is $3x^2-8x+a=0$, we can take $f_1(x)=3x^2$, $f_2(x)=8x-a$. Draw the curves $y=f_1(x)$ and $y=f_2(x)$; the real roots of $f(x)=0$ are evidently the abscissae of the points of intersection of these two curves. An iterative process for finding them may be devised as follows: select any point x_0 on the axis of x so that the value of x_0 is nearly equal to that of the abscissa of one of the points of intersection of the curves. From x_0 draw a straight line parallel to the axis of y until it meets the curve which has the slope of lesser magnitude. Suppose, for example, when $x=x_0$ that $|f_1'(x)|<|f_2'(x)|$ and that the line $x=x_0$ meets the curve $y=f_1(x)$ at the point (x_0, y_0). From this second point draw a line parallel to the axis of x until it meets $y=f_2(x)$ in the point (x_1, y_0). From the third point draw a line parallel to

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[*] For further work on iterative solutions of quadratic equations and their connection with geometry cf. O. Nicoletti, *Rend. di Palermo*, **42** (1917), p. 73.

[†] Cf. R. Ross, *Nature*, **78** (1908), p. 663.

the axis of y until it meets the curve $f_1(x) = y$ in (x_1, y_1), and from this fourth point a line parallel to the axis of x, and so on.

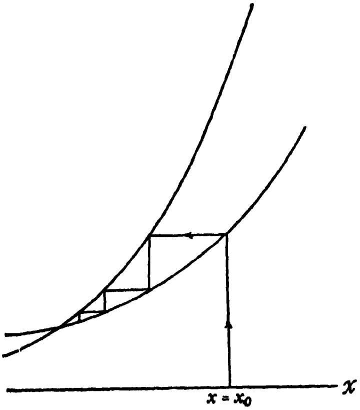


FIG. 6.

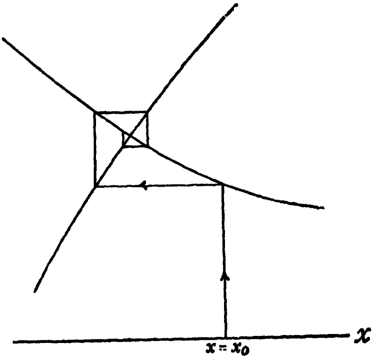
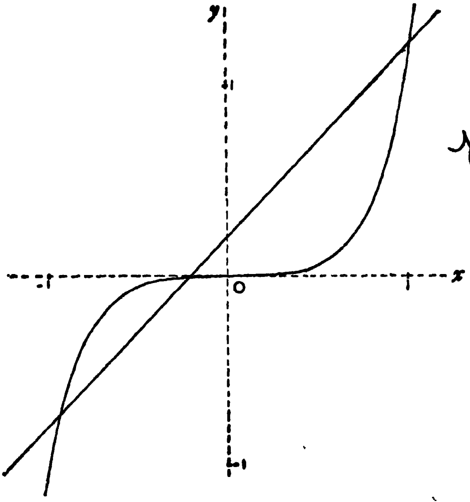


FIG. 7.

Then the abscissa of the first and second points is x_0, that of the third and fourth points is x_1, and in general x_n approaches nearer



$y = x^3$

FIG. 8.

to the point of intersection of the two curves for increasing values of n; *i.e.*, x_n converges to a root of the original equation.

There are two main types of diagram resulting from this process according as the slopes of the two curves have the same or different signs for the abscissae $x_0, x_1, x_2, \ldots$

In Fig. 6 the abscissae $x_1, x_2, x_3, \ldots$ are all on the same side of the root x and the lines approach the point of intersection of the curves in the form of a "staircase." In Fig. 7 the lines approach the intersection spirally. The *staircase* solution is obtained when the derivatives of the curves $f_1(x)$, $f_2(x)$ have the same sign near the point of intersection and the *spiral* solution occurs when those derivatives have opposite signs.

Ex. 1.—*To find the real roots of the equation*

$$x^5 - x - 0{\cdot}2 = 0.$$

The real roots are the three intersections of the curves $y = x^5$ and $y = x + 0{\cdot}2$ as shown in Fig. 8.

We can iterate to each of the three roots as follows:

For the positive root

$x = y^{\frac{1}{5}}$	$y = 0{\cdot}2 + x$
$x_0 = 1{\cdot}000$	$y_0 = 1{\cdot}200$
$x_1 = 1{\cdot}037$	$y_1 = 1{\cdot}237$
$x_2 = 1{\cdot}0434$	$y_2 = 1{\cdot}2434$
$x_3 = 1{\cdot}0445$	$y_3 = 1{\cdot}2445$
$x_4 = 1{\cdot}04472$	$y_4 = 1{\cdot}24472$

(The root is $x = 1{\cdot}0447616$.)

For the larger negative root

$x = y^{\frac{1}{5}}$	$y = 02{\cdot} + x$
$x_0 = -1{\cdot}000$	$y_0 = -0{\cdot}800$
$x_1 = -0{\cdot}956$	$y_1 = -0{\cdot}756$
$x_2 = -0{\cdot}9456$	$y_2 = -0{\cdot}7456$
$x_3 = -0{\cdot}9430$	$y_3 = -0{\cdot}7430$
$x_4 = -0{\cdot}9423$	$y_4 = -0{\cdot}7423$
$x_5 = -0{\cdot}94214$	$y_5 = -0{\cdot}74214$
$x_6 = -0{\cdot}94210$	

(The root is $x = -0{\cdot}94209$.)

For the smaller negative root

$x = y - 0{\cdot}2$	$y = x^5$
$x_0 = -0{\cdot}000$	$y_0 = -0{\cdot}000$
$x_1 = -0{\cdot}200$	$y_1 = -0{\cdot}00032$
$x_2 = -0{\cdot}20032$	$y_2 = -0{\cdot}0003225$

(Correctly to five places.)

Ex. 2.—*Find, correctly to five decimal places, the root of the equation*

$$y + \log_{10} y = 0{\cdot}5$$

by iterating the formula $y_{p+1} = 0{\cdot}5 - \log_{10} y_p$.

By inspection of a log table we see that $y_0 = 0{\cdot}68$ is an approximate value of the root. So we take

$$y_1 = 0{\cdot}5 - \log_{10} 0{\cdot}68 = 0{\cdot}6675,$$
$$y_2 = 0{\cdot}5 - \log_{10} 0{\cdot}6675 = 0{\cdot}6755.$$

As the iteration evidently furnishes values alternately less and greater than the root, we take $(y_1 + y_2)/2$ or $0{\cdot}6716$ as the next approximation y_3, and then we have

$$y_4 = 0{\cdot}5 - \log_{10} 0{\cdot}6716 = 0{\cdot}6729.$$

Take $y_5 = (y_3 + y_4)/2 = 0{\cdot}6723,$

$$y_6 = 0{\cdot}5 - \log_{10} 0{\cdot}6723 = 0{\cdot}672437,$$
$$y_7 = 0{\cdot}5 - \log_{10} 0{\cdot}672437 = 0{\cdot}672347.$$

Take $y_8 = (y_6 + y_7)/2 = 0{\cdot}672392,$

$$y_9 = 0{\cdot}5 - \log_{10} 0{\cdot}672392 = 0{\cdot}672377,$$

y_{10}, the mean of the last two values $= 0{\cdot}672385,$

$$y_{11} = 0{\cdot}5 - \log_{10} 0{\cdot}672385 = 0{\cdot}672382.$$

The root, correctly to five decimal places, is $0{\cdot}67238$.

Ex. 3.—*Find (using only Barlow's table of cubes) the smaller positive root of*

$$x^3 - 2x + 0{\cdot}5 = 0.$$

44. **The Newton-Raphson Method.**—It is evident that in iterating towards any root we are not bound to proceed by rectangular steps, as we have done in the preceding article; we might just as well have proceeded by oblique steps, as in the following method:

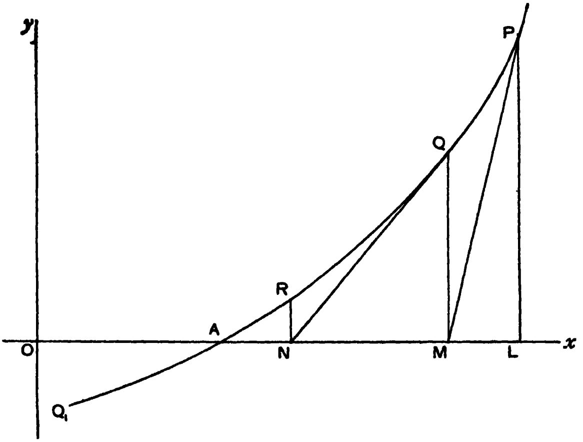


FIG. 9.

Let Q_1P be an arc of a curve $y = f(x)$, intersecting the axis of x at A, so that the abscissa of A is a root of the equation

$f(x)=0$. Suppose that the arc AP is convex to the axis of x, and that P is a point on this arc with abscissa x_0. At P draw a tangent to the curve meeting the axis of x in M and let $OM=x_1$. Let Q be the point on the curve whose abscissa is x_1, and at Q draw a tangent to meet the axis of x in N; write $ON=x_2$. Let R be the point on the curve whose abscissa is x_2, and similarly at R draw a tangent to the curve to meet the axis of x. It is evident that the points L, M, N, . . ., tend to A, or, in other words, the values x_0, x_1, x_2, x_3, . . ., form a sequence tending to the root of the equation $f(x)=0$. If, however, we start on the other side of A, where the curve is concave to the axis of x, at Q_1 say, the first step of this method carries us to the other side of A, where the arc of the curve is convex to the axis of x, after which the sequence tends to the root as before.

Now we have

$$x_0-x_1=\mathrm{ML}=\mathrm{LP}\cot \mathrm{PML}=f(x_0)/f'(x_0),$$

so
$$x_1=x_0-f(x_0)/f'(x_0),$$

and in general
$$x_{r+1}=x_r-f(x_r)/f'(x_r). \tag{1}$$

The process is therefore an iteration based on the equation (1). In substantially this form it was given by Raphson* in 1690; but the method is commonly called Newton's, because Newton had previously† suggested a nearly related process.‡

The preceding discussion is really based on two assumptions:

1. That the slope of the curve does not become zero along the arc Q_1P; *i.e.* that the equation $f'(x)=0$ has no root between x_0' and x_0, the abscissae of Q_1 and P.

2. That the curve has no point of inflexion along Q_1P.

The rule of Newton becomes more precise if we make use of the observation that we can determine which of the two abscissae x_0' and x_0 corresponds to the part of the curve which is convex towards the x-axis from the condition that *at points where the curve $y=f(x)$ is convex towards the axis of x, we have the relation*

$$f(x)f''(x)>0.$$

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* *Analysis Aequationum Universalis*, London (1690).

† Wallis' *Algebra* (1685), p. 338.

‡ The difference between Newton's process and Raphson's is that Newton calculated a set of successive equations, whose roots were the successive residuals between the above quantities x_n and the true value of the root, whereas in Raphson's form of the process this is unnecessary.

Hence we see that *if $f(x)$ has only one root between two bounds x_0' and x_0, while $f'(x)$ and $f''(x)$ are never zero between these bounds, then the Newton-Raphson process will certainly succeed if it be begun at that one of the bounds for which $f(x)$ and $f''(x)$ have the same sign.*

We might have derived Newton's method by the aid of Taylor's Theorem as follows:

Let x_0 be an approximate value of a root of the equation $f(x)=0$. Put $x=x_0+p$, where p is small. Then by Taylor's Theorem

$$0=f(x_0+p)=f(x_0)+pf'(x_0)+\text{terms involving higher powers of } p\,;$$

so approximately we have

$$p=-f(x_0)/f'(x_0),$$

and therefore

$$x=x_0-f(x_0)/f'(x_0),$$

which is Newton's formula.*

Ex. 1.—*Let the equation be*

$$x^3-2x-5=0.\dagger$$

Here it is obvious that an approximate value of the root is 2.

Taking $x_0=2$, we have

$$x_1=x_0-\frac{f(2)}{f'(2)}=2+\frac{1}{10}=2{\cdot}1.$$

Next

$$x_2=x_1-\frac{f(2{\cdot}1)}{f'(2{\cdot}1)}=2{\cdot}1-\frac{0{\cdot}061}{11{\cdot}23}=2{\cdot}1-0{\cdot}0054\text{ (nearly)}=2{\cdot}0946.$$

Now

$$\frac{f(2{\cdot}0946)}{f'(2{\cdot}0946)}=\frac{0{\cdot}0005415505}{11{\cdot}16205}=0{\cdot}000048517$$

so

$$\begin{aligned}x_3&=2{\cdot}094600000-{\cdot}000048517\\&=2{\cdot}094551483.\end{aligned}$$

The required root is 2·09455148 correctly to the first nine digits. We postpone for the present (cf. § 50) the answer to the question, how we know the number of places to which our result is correct.

Ex. 2.—*Find correctly to four decimal places the greatest root of*

$$x^3-4x^2-x+3=0.$$

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* On the formulation of conditions under which Newton's method of approximation leads to a root of an equation cf. Cauchy, *Œuvres*, Ser. 2, 4, p. 573; G. Faber, *Journ. für Math.* 138 (1910), p. 1.

† "The reason I call $x^3-2x-5=0$ a celebrated equation is because it was the one on which Wallis chanced to exhibit Newton's method when he first published it, in consequence of which *every* numerical solver has felt bound in duty to make it one of his examples. Invent a numerical method, neglect to show how it works on this equation, and you are a pilgrim who does not come in at the little wicket (*vide* J. Bunyan)," [de Morgan to Whewell, 20th January 1861].

Ex. 3.—*Find by Newton's method, correctly to six places of decimals, the root of the equation*

$$x \log_{10} x = 4{\cdot}7772393.$$

From a table of logarithms we have, by inspection, the values

$$6 \log_{10} 6 = 4{\cdot}67 \text{ and } 7 \log_{10} 7 = 5{\cdot}92,$$

so we take as a first approximation $x_0 = 6$.

The next approximation is

$$x_1 = x_0 - f(x_0)/f'(x_0).$$

Now $f(x_0) = (6 \times 0{\cdot}77815) - 4{\cdot}7772 = -0{\cdot}1083,$

$f'(x_0) = \log_{10} x_0 + \log_{10} e = 0{\cdot}778 + 0{\cdot}434 = 1{\cdot}212,$

so $1/f'(x_0) = 0{\cdot}825$ (nearly),

and $x_1 = 6{\cdot}089.$

The next approximation is

$$x_2 = x_1 - f(x_1)/f'(x_1).$$

Now $f(x_1) = (6{\cdot}089 \times 0{\cdot}7845460) - 4{\cdot}7772393 = -0{\cdot}0001387,$

so $x_2 = 6{\cdot}089 + (0{\cdot}0001387 \times 0{\cdot}825),$

or $x_2 = 6{\cdot}089114$, which is correct to six places.

45. **An Alternative Procedure.**—Instead of following Newton's rule strictly by forming $f(x_0)$ and $f'(x_0)$, etc., we may proceed in a somewhat less elaborate way as follows:

Suppose we have found (graphically or otherwise) a first approximation to the root of the equation $f(x) = 0$, which we will call a_1. Let x denote the required root of the equation and put

$$x = a_1 + \delta, \tag{1}$$

where δ is a small quantity. We now substitute this value of x in the original equation, neglecting powers of δ greater than the first. We solve the simple equation in δ so formed and denote the value obtained for δ by δ_1. Then the second approximation to the root x is $a_1 + \delta_1$. Now denote $a_1 + \delta_1$ by a_2, and write

$$x = a_2 + \delta, \tag{2}$$

and substitute this value of x in the original equation. Proceeding as before, we find an approximate value of the δ of equation (2); let it be δ_2, so $a_2 + \delta_2$ is a third approximation to the root; then we denote $a_2 + \delta_2$ by a_3, and write

$$x = a_3 + \delta, \tag{3}$$

and similarly for further operations. *The sequence* $a_1, a_2, \ldots, a_n,$ *converges to the root* x. This procedure is essentially equivalent to the Newton-Raphson process, as is evident from the connection pointed out above between the Newton-Raphson process and Taylor's Theorem (§ 44).

Ex.—Find the root of the equation

$$10x - x^3 = 3{\cdot}1462644,$$

which is near 0·3.

If $x = a_1 + \delta$ where δ is small, we have approximately

$$\delta = \frac{3{\cdot}1462644 - 10a_1 + a_1^3}{10 - 3a_1^2}.$$

Put $a_1 = 0{\cdot}3$, then

$$\delta = \frac{0{\cdot}17326}{9{\cdot}73} = 0{\cdot}17326 \times 0{\cdot}10277 = 0{\cdot}017806.$$

The next approximation is $a_2 = 0{\cdot}3178$, from which we have

$$\delta = \frac{3{\cdot}1462644 - 3{\cdot}178 + 0{\cdot}0320968}{10 - 3 \times 0{\cdot}101} = \frac{0{\cdot}0003612}{9{\cdot}697}$$

$$= 0{\cdot}000037{,}249,$$

so
$$a_3 = 0{\cdot}317837{,}249.$$

The required root of the equation is 0·3178372, correctly to seven places.

It may be remarked that the above method is theoretically applicable to complex as well as to real roots, but in the case of complex roots the numerical calculations are generally so laborious that other methods to be described later are preferable.

46. Solution of Simultaneous Equations.—As a further illustration of the method of the last section we shall find a solution of the simultaneous equations,

$$x^3 + 2y^2 = 1, \quad (1)$$

$$5y^3 + x^2 - 2xy = 4. \quad (2)$$

We first trace the curves represented by these two equations. In the first equation we find, corresponding to given values of x, the following values of y:

x	0	+1	>1	−1	−2	−3	−10·0
y	∓0·71	0	imaginary	∓1	∓2·12	∓3·74	∓22·4,

and in the second equation similarly

y	0	0·5	+1	>+1	−1	−2	−3
x	−2	−1·4	+1	imaginary	−4·16	−8·93	−15
	+2	+2·4			+2·16	+4·93	+9.

There is evidently only one real root of the equations, represented by the point G in the diagram. From the diagram we see that the ordinate of G lies between $+0{\cdot}7$ and $+0{\cdot}9$. Introducing the three values $y=0{\cdot}7$, $y=0{\cdot}8$, $y=0{\cdot}9$, into equations (1) and (2), we obtain

y.	*x by* (1).	*x by* (2).	*Difference.*
0·7	+ 0·27	− 0·95	+ 1·22
0·8	− 0·65	− 0·64	− 0·01
0·9	− 0·85	− 0·18	− 0·67,

so if in the two curves x were to vary proportionally to y we should deduce that y would be near 0·799. If we now try the three values

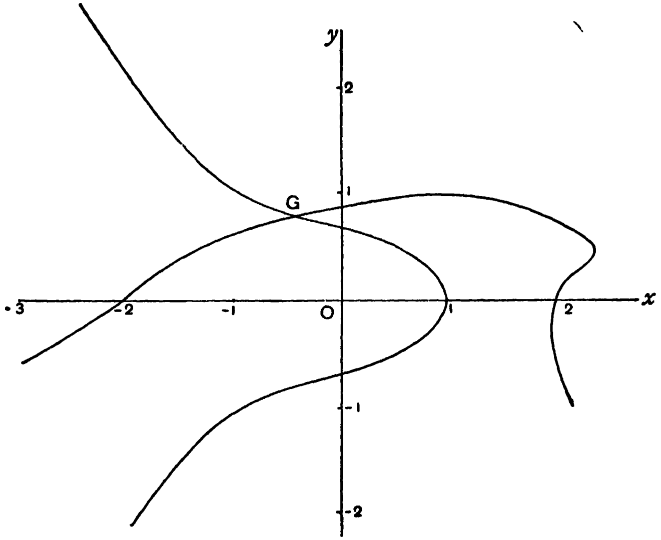

Fig. 10.

0·797, 0·798, and 0·799, this time using logs to five places of decimals, we obtain

y.	*x by* (1).	*x by* (2).	*Difference.*
0·797	− 0·6467	− 0·6534	+ 0·0067
0·798	− 0·6492	− 0·6498	+ 0·0006
0·799	− 0·6517	− 0·6460	− 0·0057,

from which it is seen that the true values are nearly

$$x = -0{\cdot}6494 \quad \text{and} \quad y = +0{\cdot}7981.$$

We therefore substitute in equations (1) and (2)

$$x = -0{\cdot}6494 + \delta x \quad \text{and} \quad y = +0{\cdot}7981 + \delta y,$$

neglecting squares of δx and δy. Using units of the third decimal place, we have

$$\begin{aligned} 1265\delta x + 3192\delta y &= -0{\cdot}0621, \\ 2895\delta x - 10853\delta y &= 0{\cdot}0959, \end{aligned}$$

whence $$\delta x = -0{\cdot}0160,\ \delta y = -0{\cdot}0131,$$

so $$x = -0{\cdot}649416,\ y = +0{\cdot}798087,$$

which is the required solution.

47. Solution of a Pair of Equations in two Unknowns by Newton's Method.

—A more formal and general treatment of the topic of the last section is the following. Let two equations be given,

$$f(x, y) = 0, \quad g(x, y) = 0,$$

from which the unknowns (x, y) are to be determined.

Let (x_0, y_0) be an approximate solution of the equations. Write

$$x = x_0 + h,$$
$$y = y_0 + k.$$

Then by Taylor's Theorem, neglecting powers of h and k above the first, we have

$$0 = f(x_0, y_0) + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0},$$

$$0 = g(x_0, y_0) + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0},$$

giving $$h = \frac{g\dfrac{\partial f}{\partial y_0} - f\dfrac{\partial g}{\partial y_0}}{\dfrac{\partial f}{\partial x_0}\dfrac{\partial g}{\partial y_0} - \dfrac{\partial f}{\partial y_0}\dfrac{\partial g}{\partial x_0}},$$

$$k = \frac{f\dfrac{\partial g}{\partial x_0} - g\dfrac{\partial f}{\partial x_0}}{\dfrac{\partial f}{\partial x_0}\dfrac{\partial g}{\partial y_0} - \dfrac{\partial f}{\partial y_0}\dfrac{\partial g}{\partial x_0}}.$$

Therefore an improved pair of values for the roots is

$$x_1 = x_0 + \frac{g\dfrac{\partial f}{\partial y_0} - f\dfrac{\partial g}{\partial y_0}}{\dfrac{\partial f}{\partial x_0}\dfrac{\partial g}{\partial y_0} - \dfrac{\partial f}{\partial y_0}\dfrac{\partial g}{\partial x_0}},$$

$$y_1 = y_0 + \frac{f\dfrac{\partial g}{\partial x_0} - g\dfrac{\partial f}{\partial x_0}}{\dfrac{\partial f}{\partial x_0}\dfrac{\partial g}{\partial y_0} - \dfrac{\partial f}{\partial y_0}\dfrac{\partial g}{\partial x_0}}.$$

These formulae may be iterated as in Newton's process for equations in one variable.

48. A Modification of the Newton-Raphson Method.

—The computations required for the Newton-Raphson method may be simplified in the following way.

Instead of

$$x_{r+1} = x_r - f(x_r)/f'(x_r),$$

we may take as the formula on which iteration is based

$$x_{r+1} = x_r - f(x_r)/f'(x_o).$$

This means that in the successive steps of the process of § 44, we replace the tangents at Q, R, . . . by lines parallel to the tangent at P. By this method we are saved the trouble of calculating $f'(x_r)$ at each stage, while the number of approximations required is practically no greater than in the Newton-Raphson method.

Ex. 1.—*To find the root of the equation*

$$f(x) \equiv x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962 = 0$$

which lies between 3 *and* 4.

Here $f(3) = -365$, $f(4) = 1110$, so by proportional parts we may take $x_0 = 3\cdot3$ as a first approximation to the root.

The next approximation is

$$x_1 = x_0 - f(x_0)/f'(x_0).$$

Now

$$f(x_0) = -65\cdot85, \quad f'(x_0) = 5x^4 + 16x^3 - 6x^2 + 20x - 2 = 1166\cdot6,$$

and $\quad 1/f'(x_0) = 0\cdot000857$ (nearly).

Therefore

$$\begin{aligned} x_1 &= 3\cdot3 + (65\cdot8 \times 0\cdot000857) \\ &= 3\cdot356. \end{aligned}$$

The next approximation is

$$x_2 = x_1 - f(x_1)/f'(x_1).$$

Instead of calculating $f'(x_1)$, we may use $f'(x_0)$ again.

Now

$$\begin{aligned} x_1 + 4 &= 7\cdot356, \\ x_1^2 + 4x_1 - 2 &= 22\cdot6867, \\ x_1^3 + 4x_1^2 - 2x_1 + 10 &= 86\cdot137, \\ x_1^4 + 4x_1^3 - 2x_1^2 + 10x_1 - 2 &= 287\cdot08, \\ x_1^5 + 4x_1^4 - 2x_1^3 + 10x_1^2 - 2x_1 - 962 &= 1\cdot44, \end{aligned}$$

and therefore $\quad f(x_1) = 1\cdot44.$

We have at once

$$\begin{aligned} x_2 &= x_1 - (1\cdot44 \times 0\cdot000857) \\ &= 3\cdot3560 - 0\cdot0012 \\ &= 3\cdot3548. \end{aligned}$$

This is correct as far as it goes, the value of the root to seven places of decimals being 3·3548487.

Ex. 2.—*Compute the root of the equation*

$$x + \log_{10} x = 0\cdot5$$

correctly to five places of decimals.

49. The Rule of False Position.—Another iteration process belonging to the same class as Newton's for finding the root of an equation is the following.

Let $f(x)=0$ be the given equation. We find (by trial or otherwise) two values a and b near the root of the equation such that $f(a)$ and $f(b)$ are opposite in sign. Let the arc CRD in the diagram denote the curve $y=f(x)$, the abscissa of R being the root of the given equation, and suppose that the equations $f'(x)=0$, $f''(x)=0$ have no root between a and b, the abscissae of C and D. The curve is therefore constantly concave to the axis of x along one of the arcs CR and RD, and constantly convex to the axis of x along the other of these arcs. Let RD be the arc which is convex to the x-axis.

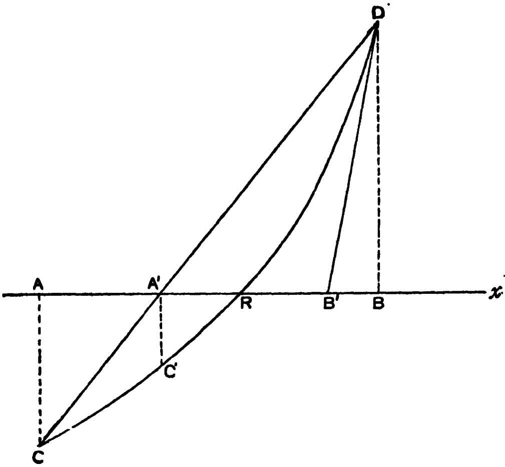

FIG. 11.

The equation of the chord CD is given by

$$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a),$$

and the abscissa a' of the point A', where the chord cuts the axis of x, is given by

$$a'=a-\frac{(b-a)f(a)}{f(b)-f(a)}. \qquad (1)$$

Then a' is evidently a closer approximation to the root of the equation than a.

We now draw the ordinate at A′ to cut the curve in C′ and, as before, we draw the chord C′D intersecting the x-axis in a point A″ which lies between A′ and R. The abscissa a'' of the point A″ is evidently a closer approximation to the value of the root than a', and its value is

$$a'' = a' - \frac{(b-a')f(a')}{f(b)-f(a')}, \tag{2}$$

and so on for further approximations to the root of the equation.

It is evident that the computation of the root may be simplified if in equation (2) we replace b by b', the abscissa of any point on the curve between R and D, as in Ex. 1 below.

Equation (1) may also be written in the form

$$a' = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

The iterative process based on this equation is known as the *rule of false position.*

The rule of false position is essentially inverse interpolation (§ 34) when differences above the first are neglected. The above iteration to the root is valid even when the initial points C, D are on the *same* side of the root R, provided the arc RD is convex to the x-axis.

Ex. 1.—*To solve the equation*

$$x^3 - 2x - 5 = 0.$$

Here we can take $a = 2$, $b = 3$, and

$$a' = 2 - \frac{(3-2)f(2)}{f(3)-f(2)} = 2 + \frac{1}{17},$$

i.e., $a' = 2{\cdot}06$.

We now take $a' = 2{\cdot}06$, $b' = 2{\cdot}10$, then

$$a' = 2{\cdot}06 - \frac{(2{\cdot}10 - 2{\cdot}06)\,(-0{\cdot}378184)}{0{\cdot}061000 + 0{\cdot}378184} = 2{\cdot}0944\ \ldots,$$

i.e., $a' = 2{\cdot}094$.

Instead of taking $b' = 2{\cdot}10$ as before, we may take $b' = 2{\cdot}096$ since $f(2{\cdot}096)$ is positive. We have

$$a'' = 2{\cdot}094 - \frac{0{\cdot}002\,(-0{\cdot}006153416)}{0{\cdot}016180736 + 0{\cdot}006153416} = 2{\cdot}0945510$$

The required root is 2·094551 correctly to seven significant figures.

When the rule of false position is written in the form

$$\frac{f(b)-f(a)}{b-a}=-\frac{f(a)}{a'-a}=-\frac{f(b)}{a'-b} \qquad (3)$$

we may express the rule as follows:*

Assume two numbers as near the true root as possible, and find the error arising from the substitution of each of these quantities instead of the unknown quantity in the proposed equation; then as the difference between the two errors is to the difference of the assumed numbers so is either error to the correction of the corresponding assumed number.

Assuming this new value a' instead of a and another quantity b' differing from a' only by one unit in the last place so that b' is greater or less than a' according as a' is found too small or too great, we then find a new approximation a'' and so on to any required degree of accuracy.

Ex. 2.—*Solve in this way the above equation*

$$x^3-2x-5=0.$$

50. Combination of the Methods of § 44 and § 49.

—It was remarked by Dandelin † that by combining Newton's rule with the rule of false position we are in possession of a method of solving equations in which upper and lower bounds to the value of the root are obtained at every stage of the process, so that any digits common to the two bounds certainly belong to the correct value of the root.

Thus using the figure of the last section, and still assuming that RD is the arc which is convex to the x-axis, we draw as before the chord CD to cut the axis of x in the point A′ whose abscissa a' is given by the equation

$$a'=a-\frac{(b-a)f(a)}{f(b)-f(a)}\ . \ . \ ., \qquad (1)$$

where a, b are the abscissae of the points C, D. Then a' is a closer approximation to the root of the equation than a. We now draw the tangent at D to cut the axis of x in B′. If b is the abscissa of D, then b', the abscissa of B′, is given by Newton's rule in the form

$$b'=b-\frac{f(b)}{f'(b)}\ . \ . \ ., \qquad (2)$$

and b' is a closer approximation to the root of the equation than b. It is evident that the root of the equation must lie between the bounds a' and b'.

---

* Barlow's *Mathematical Tables* (1814), p. xxxvi.

† *Mém. de l'Acad. Royale de Bruxelles*, 3 (1826), p. 30.

We may now operate on a' and b' as on a and b, to find two new bounds of the root, namely,

$$a'' = a' - \frac{(b'-a')f(a')}{f(b')-f(a')} \quad \text{and} \quad b'' = b' - \frac{f(b')}{f'(b')},$$

which envelop the root more closely than a' and b', and so on for further approximations to the root.

It may be remarked here that at the conclusion of the process it is best to take, as our final value for the root of the equation, the arithmetical mean of the pair of values last calculated for a and b.

Ex.—Consider again the equation

$$f(x) \equiv x^3 - 2x - 5 = 0,$$

for which $f(x)$ has a root between 2 and 2·1, while $f''(x)$ has no root between these limits.

Now $\qquad f(2) \times f''(2) < 0$ and $f(2\cdot 1) \times f''(2\cdot 1) > 0$,

so if $a = 2$, $b = 2\cdot 1$, the curve $y = f(x)$ is convex to the axis when $x = b$.

The next approximations are:

$$\begin{aligned} b &= 2\cdot 1 - \frac{f(2\cdot 1)}{f'(2\cdot 1)} = 2\cdot 1 - \frac{0\cdot 061}{11\cdot 230} \\ &= 2\cdot 09457, \\ a' &= 2 - \frac{f(2) \times (2\cdot 1 - 2)}{f(2\cdot 1) - f(2)} \\ &= 2 + \frac{1}{10\cdot 61} = 2\cdot 0943. \end{aligned}$$

Comparing b' and a', we see that 2·094 are digits of the true root.

Now take $\qquad a' = 2\cdot 0943, \quad b' = 2\cdot 0946$

and we get

$$\begin{aligned} a'' &= a' - \frac{f(a')\,(b'-a')}{f(b')-f(a')} = 2\cdot 0943 + \frac{0\cdot 000000{,}841949{,}4579}{0\cdot 003348{,}048729} \\ &= 2\cdot 094551{,}475, \\ b'' &= b' - f(b')/f'(b') = 2\cdot 0946 - \frac{0\cdot 000541{,}550536}{11\cdot 162047{,}48} \\ &= 2\cdot 094551{,}483, \end{aligned}$$

so the first eight digits of the root are 2·094551,4.

The arithmetic mean between a'' and b'' is

$$2\cdot 094551{,}48,$$

which gives the first nine digits correctly.

51. **Solutions of Equations by the use of the Calculus of Differences.**—We have already seen in § 34 how equations may be solved by *inverse interpolation.* We shall now show by an example how *divided differences* (Chap. II.) may be applied for this purpose.

Suppose it is required to solve the equation

$$y \equiv x^3 + 3x^2 - 12x - 10 = 0.$$

The coefficients -10, -12, 3, 1, are of course the divided differences of y for the set of coincident arguments 0, 0, 0, 0, so we can write down part of a table of divided differences thus:

x.	y.			
0		–		
0	–		–	
		–		1
0	–		3	
		-12		
0	-10			

and this we shall now extend downwards. The third differences of y are constant, and therefore if we take as the next argument $x = 2$, we have

x.	y.			
0	–		–	
		–		1
0	–		3	
		-12		1
0	-10		5	
		-2		
2	-14			

Here the number 5, which is the new divided difference of the 2nd order, is obtained from the equation $(p-3)/2 = 1$ giving $p = 5$: then the new difference of the first order is obtained from $(q+12)/2 = 5$ giving $q = -2$; and lastly the value of the function corresponding to the argument $x = 2$ is obtained from $(r+10)/2 = -2$ giving $r = -14$. In this way we construct the following table of divided differences, taking arguments suggested by the sequence of values of y already obtained:

x.	y.			
0	−10		3	
		−12		1
0	−10		5	
		−2		1
2	−14		7·5	
		16·75		1
2·5	−5·625		10·2	
		23·89		1
2·7	−0·847		10·9	
		26·07		1
2·7	−0·847		11·1	
		26·07		1
2·7	−0·847		11·13	
		26·4039		1
2·73	−0·054883		11·16	
		26·7387		1
2·73	−0·054883		11·19	
		26·7387		1
2·73	−0·054883		11·192	
		26·761084		1
2·732	−0·0013608		11·194	

Having found that 2·7 is near the root, since y is comparatively small, we repeat the interpolation and thus obtain, (§ 16)

$$y = -0{\cdot}847 + 26{\cdot}07(x - 2{\cdot}7) + 11{\cdot}1(x - 2{\cdot}7)^2 \text{ nearly,}$$

giving $x - 2{\cdot}7 = 0{\cdot}03$ approximately; so we take 2·73 as our next approximation. We continue this method until the approximations are sufficiently accurate for our purpose.

If we require the approximation correctly to four digits, we have $x = 2{\cdot}732$ and the next digit is found to be 0, so the required root is 2·732.

Having obtained this root, we proceed to find approximate values for the remaining roots of the equation. Thus we first transform y into a polynomial in $(x - 2{\cdot}732)$,

2·73	− 0·054883		11·192	
		26·761084		1
2·732	− 0·001361		11·194	
		26·783472		1
2·732	− 0·001361		11·196	
		26·783472		
2·732	− 0·001361			

so

$$y = 26{\cdot}783(x - 2{\cdot}732) + 11{\cdot}196(x - 2{\cdot}732)^2 + (x - 2{\cdot}732)^3 \text{ (nearly).}$$

We may now divide out the factor $x - 2{\cdot}732$ and solve the quadratic so formed,

$$26{\cdot}783 + 11{\cdot}196\ (x - 2{\cdot}732) + (x - 2{\cdot}732)^2 = 0.$$

The remaining roots are found to be

$$\begin{cases} \quad x = \ -5{\cdot}000 \\ \text{and } x = \ -0{\cdot}732, \end{cases}$$

which are correct to four significant figures.

(The actual roots of the above equation are 2·7320508, − 0·7320508, and − 5.)

Ex.—Find correctly to four significant figures the root of the equation

$$x^3 - 9x^2 + 23x - 14 = 0,$$

which is between + 4 *and* + 5.

52. **The Method of Daniel Bernoulli.**—In 1728 Daniel Bernoulli* devised a method wholly different in principle from any which were then known. Though hardly now of first-rate importance, it is interesting and worthy of mention.

Let it be required to solve the equation

$$a_0x^n + a_1x^{n-1} + \ .\ .\ .\ + a_n = 0. \qquad (1)$$

Consider the difference equation

$$a_0y\ (t + n) + a_1y\ (t + n - 1) + \ .\ .\ .\ + a_ny\ (t) = 0. \qquad (2)$$

The solution of (2) is known to be

$$y(t) = w_1\, x_1{}^t + w_2\, x_2{}^t + \ .\ .\ .\ + w_n\, x_n{}^t,\ .\ .\ . \qquad (3)$$

where w_1, w_2, . . ., w_n are arbitrary functions of t of period 1, and x_1, . . ., x_n are the roots of equation (1).

---

* *Commentarii Acad. Sc. Petropol.* III. (1732); Cf. Euler, *Introductio in Analy. Inf.* I. cap. XVII.; Lagrange, *Résolution des équations numériques*, Note 6.

If $|x_1|$ is greater than the modulus of any other root, the first term on the right in (3) becomes very large compared with the other terms when t is large; and therefore we have

$$x_1 = \underset{t\to\infty}{\mathrm{Lt}} \frac{y(t+1)}{y(t)}.$$

This leads at once to Bernoulli's rule, which is as follows:

In order to find the absolutely greatest root of the equation (1), *we take any arbitrary values for* $y(0), y(1), y(2), \ldots, y(n-1)$; *from these by repeated application of equation* (2) *we calculate in succession the values of* $y(n), y(n+1), y(n+2), \ldots$. *The ratio of two successive members of this sequence tends in general* to a limit, which is the absolutely greatest root of the equation* (1).

Ex. 1.—*Find the absolutely greatest root of*

$$x^5 + 5x^4 - 5 = 0.$$

Consider the difference equation

$$y(t+5) + 5y(t+4) - 5y(t) = 0, \tag{1}$$

and write down arbitrarily the values $y(0)=0, y(1)=0, y(2)=0, y(3)=0, y(4)=1$. By means of equation (1), we have the following values of $y(5)$, $y(6)$, etc.:

t	5	6	7	8	9	10	11	12	13
$y(t)$	−5	25	−125	625	−3,120	15,575	−77,750	388,125	−1,937,500

Also

$$y(14) = -5y(13) + 5y(9),$$

so

$$\frac{y(14)}{y(13)} = -5 + \frac{5y(9)}{y(13)} = -5 + \frac{156}{19375}$$

$$= -4{\cdot}991948.$$

The absolutely greatest root of the equation is therefore given by this method as approximately $-4{\cdot}991948$; this value is as a matter of fact correct to the last digit. If we had stopped earlier, we might have obtained, *e.g.*:

$$\frac{y(12)}{y(11)} = \frac{388{,}125}{-77{,}750} = -4{\cdot}99196,$$

which is in error only in the sixth significant digit.

Ex. 2.—*Find the smallest root of*

$$z^3 - 6z^2 + 9z - 1 = 0$$

correctly to seven places by Bernoulli's method.

(*Note.*—Put $z = \frac{1}{x}$ and solve for x.)

---

* If the ratio does not tend to a limit, but oscillates, the root of greatest modulus is one of a pair of conjugate complex roots.

53. The Ruffini-Horner Method.* — As we have mentioned in § 41, the method of Vieta was the common method of solving algebraic equations until it was superseded by Newton's method of approximation. The reason why the Newtonian approximation was found to be less laborious than its predecessor was that the earlier method contained no provision for making the steps of one part of the process facilitate those which succeed.

In 1819 W. G. Horner discovered a rule † for performing the computations necessary in Vieta's method, by which that method was made preferable to the Newtonian, and so restored to favour.

Horner's contribution was essentially a convenient numerical process for computing the coefficients of the equation whose roots differ by a given constant from the roots of a given equation. We shall first consider this process.

Let $f(x) = 0$ be a given equation where $f(x)$ is a polynomial in x, and let it be required to find the equation whose roots are the roots of this equation, each diminished by r. This equation will be $f(x+r) = 0$ or

$$0 = f(r) + xf'(r) + \frac{x^2}{2!}f''(r) + \frac{x^3}{3!}f'''(r) + \dots \qquad (1)$$

The expressions $f(r), f'(r), f''(r)/2!, f'''(r)/3!, \dots$ may now be found in the following way. Suppose for example that

$$f(x) \equiv Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F.$$

Write down the coefficients A, B, C, D, . . ., F in a horizontal row and form from them the following scheme in which a new letter below a line stands for the sum of the two immediately above it, *e.g.* $P = Ar + B$.

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* Ruffini, *Sopra la determinazione delle radici*, Modena (1804); and *Memorie di Mat. e di Fis. della Soc. Italiana delle Scienze*, Verona (1813).

Horner, *Phil. Trans.* (1819), Part I. p. 308; and *The Mathematician*, 1 (1845), p. 109.

† So far as the cube root is concerned, it had been given previously by Alexander Ingram in the Appendix to his edition of Hutton's *Arithmetic* (Edinburgh, 1807). A method based on the same principles had been discovered in the thirteenth century by the Chinese mathematicians.

$$\begin{array}{cccccc}
A & B & C & D & E & F \\
 & \underline{Ar} & \underline{Pr} & \underline{Qr} & \underline{Rr} & \underline{Sr} \\
 & P & Q & R & S & \omega \\
 & \underline{Ar} & \underline{Tr} & \underline{Ur} & \underline{Vr} & \\
 & T & U & V & \chi & \\
 & \underline{Ar} & \underline{Wr} & \underline{Xr} & & \\
 & W & X & \psi & & \\
 & \underline{Ar} & \underline{Yr} & & & \\
 & Y & \phi & & & \\
 & \underline{Ar} & & & & \\
 & \theta & & & &
\end{array}$$

It is seen at once that the values of θ, ϕ, ψ, χ, ω, thus obtained at the feet of the columns, have respectively the values

$$\theta = 5Ar + B = \frac{1}{4!}f^{(\text{iv})}(r),$$

$$\phi = 10Ar^2 + 4Br + C = \frac{1}{3!}f'''(r),$$

$$\psi = 10Ar^3 + 6Br^2 + 3Cr + D = \frac{1}{2!}f''(r),$$

$$\chi = 5Ar^4 + 4Br^3 + 3Cr^2 + 2Dr + E = f'(r),$$

$$\omega = Ar^5 + Br^4 + Cr^3 + Dr^2 + Er + F = f(r),$$

and therefore the equation (1) whose roots are the roots of the equation $f(x) = 0$, each diminished by r, becomes

$$0 = Ax^5 + \theta x^4 + \phi x^3 + \psi x^2 + \chi x + \omega. \tag{2}$$

The above scheme therefore enables us to find readily the equation whose roots are the roots of a given equation, each diminished by a given number.

This process is first applied in order to diminish a root of the proposed equation by its first digit; then it is again applied in order to diminish the corresponding root of the resulting equation by its first digit, which is the second digit of the required root of the original equation; then again in diminishing the root of the equation last obtained by its first digit, which is the third digit of the required root; and so on.

Note that $r - \frac{\omega}{\chi}$ is the same as $r - \frac{f(r)}{f'(r)}$, and this is the quantity which would be given by the Newton-Raphson method as an improved approximation to the root, after r had been found as a first approximation. This property is used at each stage of the process in order to obtain the next digit of the root; in fact, we use the two last terms $\chi x + \omega$ of equation (2) to suggest the next digit.

We shall apply this method *to find the smallest positive root of the equation*

$$x^3 - 4x^2 + 5 = 0.$$

By means of a graph of the curves $y = x^3$, $y = 4x^2 - 5$ it is readily seen that the required root lies between 1 and 2, so we shall first diminish the roots of the given equation by unity.

$$\begin{array}{rrrr}
1 & -4 & 0 & 5(1 \\
 & \underline{1} & \underline{-3} & \underline{-3} \\
 & -3 & -3 & 2 \\
 & \underline{1} & \underline{-2} & \\
 & -2 & -5 & \\
 & \underline{1} & & \\
 & -1 & &
\end{array}$$

The equation whose roots are the roots of the original equation diminished by 1 is therefore

$$x^3 - x^2 - 5x + 2 = 0. \qquad (1)$$

We now form the equation whose roots are ten times the roots of this equation; it is

$$x^3 - 10x^2 - 500x + 2000 = 0.$$

We want the root of this equation which lies between 1 and 10, the other roots being numerically greater than 10. It is found to be between 3 and 4,* so we diminish the roots by 3, thus:

$$\begin{array}{rrr}
-10 & -500 & 2000(3 \\
\underline{3} & \underline{-\ \ 21} & \underline{-1563} \\
-\ \ 7 & -521 & 437 \\
\underline{3} & \underline{-\ \ 12} & \\
-\ \ 4 & -533 & \\
\underline{3} & & \\
-\ \ 1 & &
\end{array}$$

so the transformed equation is

$$x^3 - x^2 - 533x + 437 = 0. \qquad (2)$$

---

* By the principles explained above, viz. that the two last terms of this equation have the chief influence in determining the root, the two first terms being evidently small compared with the two last terms, when x lies between 1 and 10.

Multiplying the roots of this equation by 10, it becomes

$$x^3 - 10x^2 - 53300x + 437000 = 0.$$

This equation has a root between 8 and 9 (as is seen by the inspection of the last two terms since $437000/53300 = 8{\cdot}19\ldots$), so we diminish the roots by 8.

$$\begin{array}{cccc}
1 & -10 & -53300 & 437000(8 \\
 & 8 & -16 & -426528 \\
\cline{2-4}
 & -\ 2 & -53316 & 10472 \\
 & 8 & 48 & \\
\cline{2-3}
 & 6 & -53268 & \\
 & 8 & & \\
\cline{2-2}
 & 14 & &
\end{array}$$

The transformed equation is therefore

$$x^3 + 14x^2 - 53268x + 10472 = 0, \tag{3}$$

and, multiplying the roots by 10, it becomes

$$x^3 + 140a^2 - 5326800x + 10472000 = 0, \tag{4}$$

for which an approximate value of the root is

$$\frac{10472000}{5326800} \text{ or } 1{\cdot}9659.$$

Thus, finally, we obtain for the required root of the original equation the value

$$1{\cdot}3819659,$$

which is in error only by one unit in the seventh place of decimals.

We may note that approximate values of the other two roots may be obtained from the equation at this stage in a very simple fashion. For if a cubic equation has two roots M and N, which are numerically very large in comparison with the third root ϵ, the cubic is nearly

$$x^3 - (\mathrm{M} + \mathrm{N})x^2 + \mathrm{MN}x - \mathrm{MN}\epsilon = 0.$$

The other two roots of the cubic (4) will therefore be approximately the roots of the quadratic

$$x^2 + 140x - 5326800 = 0,$$

which are $x = \ \ 2239{\cdot}05$

and $x = -2379{\cdot}05,$

and therefore the two corresponding roots of the original cubic are these values divided by 1000 and increased by 1·38 (the part of the root already found), viz.

$x = -0{\cdot}999$

and $x = \ \ 3{\cdot}619.$

These are in error by one unit in the third place of decimals.

We may note that *Horner's Process is really the formation of a table of divided differences.* Thus taking the last example again, suppose it is required to find the equation whose roots are the roots of

$$X = x^3 - 4x^2 + 5 = 0,$$

each diminished by unity. We note that the coefficients 5, 0, -4, 1 are the divided differences of X for the set of coincident arguments 0, 0, 0, 0, so we can write down the following table of divided differences:

x	X			
				1
0	5		-4	
		0		1
0	5		-3	
		-3		1
1	2		-2	
		-5		1
1	2		-1	
		-5		
1	2			

whence, by Newton's formula for interpolation with repeated arguments (§ 16) we have

$$X \equiv 2 - 5(x-1) - (x-1)^2 + (x-1)^3$$

or

$$X \equiv 2 - 5y - y^2 + y^3,$$

which is equation (1) above.

Therefore the equation whose roots are ten times the roots of the reduced equation is

$$X \equiv 2000 - 500x - 10x^2 + x^3 = 0.$$

Now diminish the roots by 3. This is done by the difference table

x	X			
				1
0	2000		-10	
		-500		1
0	2000		-7	
		-521		1
3	437		-4	
		-533		1
3	437		-1	
		-533		
3	437			

so that $$X \equiv 437 - 533(x-3) - (x-3)^2 + (x-3)^3,$$

which gives us precisely equation (2) above; and so on.

Ex. 1.—*Find to six significant digits the positive root of the equation*

$$x^3 + 3x^2 - 12x - 10 = 0$$

by performing three Horner's transformations and then approximating.

First, diminish the roots by 2,

$$\begin{array}{rrrr}
1 & 3 & -12 & -10 \\
 & \underline{2} & \underline{10} & \underline{-\ 4} \\
 & 5 & -\ 2 & -14 \\
 & \underline{2} & \underline{14} & \\
 & 7 & 12 & \\
 & \underline{2} & & \\
 & 9 & &
\end{array}$$

Therefore the new equation is

$$x^3 + 9x^2 + 12x - 14 = 0.$$

Multiply the roots by 10,

$$x^3 + 90x^2 + 1200x - 14000 = 0.$$

Diminish the roots by 7,

$$\begin{array}{rrrr}
1 & 90 & 1200 & -14000 \\
 & \underline{7} & \underline{679} & \underline{13153} \\
 & 97 & 1879 & -\ \ 847 \\
 & \underline{7} & \underline{728} & \\
 & 104 & 2607 & \\
 & \underline{7} & & \\
 & 111 & &
\end{array}$$

Therefore the new equation is

$$x^3 + 111x^2 + 2607x - 847 = 0.$$

Multiply the roots by 10,

$$x^3 + 1110x^2 + 260700x - 847000 = 0.$$

Diminish the roots by 3,

$$\begin{array}{rrrr}
1 & 1110 & 260700 & -847000 \\
 & \underline{3} & \underline{3339} & \underline{792117} \\
 & 1113 & 264039 & -\ \ 54883 \\
 & \underline{3} & \underline{3348} & \\
 & 1116 & 267387 & \\
 & \underline{3} & & \\
 & 1119 & &
\end{array}$$

Therefore the new equation is

$$x^3 + 1119x^2 + 267387x - 54883 = 0.$$

Moreover, $$\frac{54883}{267387} = 0{\cdot}205 \text{ (nearly)},$$

so that the required root is 2·73205.

To get a better approximation, we have from the last equation, approximately,

$$\begin{aligned} x &= \frac{54883}{267387} - \frac{1119}{267387}x^2 \\ &= 0{\cdot}205257 - (0{\cdot}0041849)(0{\cdot}205)^2 \\ &= 0{\cdot}205257 - 0{\cdot}000176 \\ &= 0{\cdot}205081, \end{aligned}$$

so the required root is 2·732050,8.

Ex. 2.—*Find correctly to seven significant digits that root of the equation*

$$x^3 - 9x^2 + 23x - 14 = 0$$

which is between + 4 *and* + 5, *by performing three Horner's transformations and then approximating.*

Ex. 3.—*Find by Horner's method a root of the equation* *

$$x^3 - 2x = 5.$$

54. The Root-squaring Method of Dandelin, Lobachevsky, and Graeffe.—We shall next consider a method of solving equations which was suggested independently by Dandelin † in 1826, Lobachevsky ‡ in 1834, and Graeffe § in 1837, and which is frequently of great use, especially in the case of equations possessing complex roots. It has the advantage (when performed completely) of finding all the roots at once and of not requiring any preliminary determination of their approximate position. Its principle is to form a new equation whose roots are some high power of the roots of the given equation: suppose we say the 128th power, so that if the roots of the given equation are $x_1, x_2, x_3, \ldots$, then the roots of the new equation are $x_1^{128}, x_2^{128}, x_3^{128}, \ldots$ These numbers are *widely separated*; thus if x_1 were twice x_2, then x_1^{128} would be more than 10^{38} times x_2^{128}, and, as we shall see, an equation whose roots are very widely separated can be solved at once numerically.

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* A pupil of De Morgan by Horner's method found the root of Wallis's well-known example $x^3 - 2x = 5$ to 51 places to be

$x = 2{\cdot}094\ 551,\ 481\ 542,\ 326\ 591,\ 482\ 386,\ 540\ 579,\ 302\ 963,\ 857\ 306,\ 105\ 628\ 239.$

This was subsequently extended to 101 decimal places; cf. *The Mathematician*, **3** (1850), p. 290.

† *Mém. de l'Acad. Royale de Bruxelles*, **3** (1826), p. 48.

‡ *Algebra* or *Calculus of Finites*, Kazàn (1834), § 257.

§ *Auflösung der höheren numerischen Gleichungen*, Zürich (1837). The method was perfected by Brodetsky and Smeal, *Proc. Camb. Phil. Soc.*, **22** (1924), p. 83.

Suppose it is required to solve the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_n = 0. \qquad (1)$$

In this equation we shall denote the roots, which will for the present be assumed to be real and unequal, by $-a$, $-b$, $-c$, $-d$, . . ., the order being that of descending numerical magnitude, so that $|a|>|b|>|c|>|d|$ . . . The values a, b, c, d, . . ., which are the roots of the equation reversed in sign, will be called the *Encke** roots. Equation (1) may now be written in the form

$$x^n + [a]x^{n-1} + [ab]x^{n-2} + [abc]x^{n-3} + \ldots = 0, \qquad (2)$$

where $[a]$ denotes $a+b+c+d+\ldots$, that is, the sum of the Encke roots, $[ab]$ denotes $ab+ac+bc+\ldots$, the sum of the products of the Encke roots taken two at a time, and so on. If, moreover, we denote the sum $a^m+b^m+c^m+d^m+\ldots$ by $[a^m]$, then the equation whose roots are the mth powers of the roots of the given equation will evidently, if m is even, be

$$x^n + [a^m]x^{n-1} + [a^mb^m]x^{n-2} + [a^mb^mc^m]x^{n-3} + \ldots = 0. \qquad (3)$$

The problem before us is to construct the equation (3) when the equation (2) is given, and m is some prescribed number. In practice m is a large number in the equation (3) which is ultimately formed, but we do not attempt to construct this equation at a single step of the process; instead of this we first take $m=2$, that is to say, we form a new equation whose Encke roots are the squares of the Encke roots of the original equation; then, having done this, we repeat the process, forming a new equation whose Encke roots are the squares of the Encke roots of the equation just obtained—that is, the 4th powers of the Encke roots of the original equation—and so on.

Thus our immediate problem is to construct the equation (3) when equation (2) is given and m has the value 2. This we do in the following way. Rearrange the given equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_n = 0, \qquad (1)$$

so that the terms containing the even powers of x are on one side of the equation, and the terms containing the odd powers are on the other side. Squaring both sides of the resulting equation, we have at once

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* Encke, *Journal für Math.* **22** (1841), p. 193.

$$(x^n + a_2x^{n-2} + a_4x^{n-4} + \dots)^2 = (a_1x^{n-1} + a_3x^{n-3} + \dots)^2,$$

or, putting $-x^2 = y$,

$$y^n + (a_1^2 - 2a_2)y^{n-1} + (a_2^2 - 2a_1a_3 + 2a_4)y^{n-2} + \dots = 0. \qquad (4)$$

Since the roots of equation (1) are $-a$, $-b$, $-c$, . . ., and $-x^2 = y$, the roots of equation (4) are $-a^2$, $-b^2$, $-c^2$, . . .; that is to say (writing x in place of y), *the equation whose Encke roots are the squares of the Encke roots of* (1) *is*

$$x^n + \left.\begin{matrix} a_1^2 \\ -2a_2 \end{matrix}\right\} x^{n-1} + \left.\begin{matrix} a_2^2 \\ -2a_1a_3 \\ +2a_4 \end{matrix}\right\} x^{n-2} + \left.\begin{matrix} a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_6 \end{matrix}\right\} x^{n-3} + \left.\begin{matrix} a_4^2 \\ -2a_3a_5 + 2a_2a_6 \\ -2a_1a_7 \\ +2a_8 \end{matrix}\right\} x^{n-4} + \dots = 0. \qquad (5)$$

The law of formation of the coefficients in equation (5) may be stated thus: *The coefficient of any power of x is formed by adding to the square of the corresponding coefficient in the original equation the doubled product of every pair of coefficients which stand equally far from it on either side, these products being taken with signs alternately negative and positive.*

Having thus formed the equation whose Encke roots are the squares of the Encke roots of (1), we repeat the process, thus obtaining an equation whose Encke roots are the squares of the Encke roots of (5), that is to say, the 4th powers of the Encke roots of (1). The next stage yields an equation whose Encke roots are the 8th powers of the Encke roots of (1), and so on.

Now consider the equation which is obtained when the process has been repeated several times, so that we have calculated the equation (3) where m is (say) 64 or 128 or 256. Since a is numerically larger than b, therefore a^m is enormously larger than b^m or c^m or d^m . . ., and thus the sum $[a^m]$ bears to its first term a^m a ratio which is very near to unity. Similarly $[a^mb^m]$ bears to its first term a^mb^m a ratio which is very near to unity, and so on.

If, then, the ratio of $[a^m]$ to a^m is $1 + \epsilon$, where ϵ is small, we have

$$\log [a^m] = m \log |a| + \log (1 + \epsilon)$$

or
$$\log |a| = \frac{1}{m} \log [a^m] - \frac{1}{m} \log (1 + \epsilon).$$

Since we have calculated $[a^m]$, the right-hand side of this equation is known except for the small quantity $\frac{1}{m} \log (1 + \epsilon)$, which we may neglect; and thus $|a|$ is determined.

Next, since $[a^m b^m] = a^m b^m (1 + \gamma)$, where γ is small, we have
$$\log |ab| = \frac{1}{m} \log [a^m b^m], \text{neglecting } \frac{1}{m} \log (1 + \gamma),$$
and therefore
$$\log |b| = \frac{1}{m} \log [a^m b^m] - \frac{1}{m} \log [a^m],$$
which determines the modulus of the second root b, and so on.

In solving an equation by this method, it is all-important to know when to stop. Obviously the time to stop is when another doubling of m would give a result not different (in the digits we wish to include) from the result which would be obtained by stopping at once; that is to say, when the coefficients $[a^{2m}]$, $[a^{2m} b^{2m}]$, . . . of the new equation are practically nothing but the squares of the corresponding coefficients $[a^m]$, $[a^m b^m]$, . . . in the equation already obtained.

Ex. 1.—*To solve the equation*
$$x^3 + 9x^2 + 23x + 14 = 0.$$

The equation whose Encke roots are the squares of the Encke roots of this equation is
$$x^3 + 35x^2 + 277x + 196 = 0.$$

The equation whose Encke roots are the squares of these is
$$x^3 + 671x^2 + 63009x + 38416 = 0,$$
and proceeding in this way we obtain the following equations:

$$x^3 + 324223x^2 + 3{\cdot}91858 \times 10^9 x + 1{\cdot}475789 \times 10^9 = 0.$$

$$x^3 + 9{\cdot}72834 \times 10^{10} x^2 + 1{\cdot}535431 \times 10^{19} x + 2{\cdot}177953 \times 10^{18} = 0.$$

$$x^3 + 9{\cdot}43335 \times 10^{21} x^2 + 2{\cdot}357548 \times 10^{38} x + (2{\cdot}177953 \times 10^{18})^2 = 0.$$

$$x^3 + 8{\cdot}898762 \times 10^{43} x^2 + (2{\cdot}357548 \times 10^{38})^2 x + (2{\cdot}177953 \times 10^{18})^4 = 0.$$

Stopping at this stage, we have

$$\log (a^{64}) = \log (8{\cdot}898762 \times 10^{43}) = 43{\cdot}9493296,$$

whence

$$\log |a| = 0{\cdot}6867083,$$

so the numerically greatest root is ± 4·860806.

Also

$$\log a^{64}b^{64} = 2 \log (2{\cdot}357548 \times 10^{38})$$
$$= 76{\cdot}7449212,$$

so

$$\log |b| = \tfrac{1}{64}(76{\cdot}7449212 - 43{\cdot}9493296) = 0{\cdot}5124311$$
$$= \log 3{\cdot}2541015.$$

The next root is therefore ± 3·2541015.

Lastly,

$$\log a^{64}b^{64}c^{64} = 4 \log (2{\cdot}177953 \times 10^{18}),$$

so

$$\log |c| = \tfrac{1}{64}(73{\cdot}3521936 - 76{\cdot}7449212) = \bar{1}{\cdot}9469886$$
$$= \log 0{\cdot}8850924.$$

The numerically smallest root is therefore ± 0·8850924.

A rough graph shows that all the roots are negative; they are therefore

$$-4{\cdot}860806, \quad -3{\cdot}2541015, \quad \text{and} \quad -0{\cdot}8850924.$$

We do not usually require to compute each of the roots of an equation to the same degree of accuracy, although we may require to know the approximate values of all the roots. We shall therefore give here a more rapid way of computing the roots to two or three significant figures, in which we use a 4-place table of squares, such as Barlow's Tables, and 4-place tables of logarithms. We shall denote by p^m the equation whose roots are minus the mth powers of the roots of the original equation, and we shall write, *e.g.*, 1896^p for the number $1{\cdot}896 \times 10^p$.

Ex. 2.—*To solve by this method the equation of Ex.* 1 *:*

$$x^3 + 9x^2 + 23x + 14 = 0.$$

We first arrange the coefficients of the equation in order :

x^3.	x^2.	x.	x^0.
1	9000	2300^1	1400^1

Squaring these coefficients and adding the doubled products as in equation (5), we compute p^2, then p^4, p^8, and so on, arranging the values of the successive equations in tabular form, thus :

	1	8100^{1}	5290^{2}	1960^{2}
		-4600	-2520	
p^{2}	1	3500^{1}	2770^{2}	1960^{2}
	1	1225^{3}	7673^{4}	3842^{4}
		-0554	-1372	
p^{4}	1	0671^{3}	6301^{4}	3842^{4}
	1	4502^{5}	3970^{9}	1476^{9}
		-1260	-0052	
p^{8}	1	3242^{5}	3918^{9}	1476^{9}
	1	1051^{11}	1535^{19}	2179^{18}
		-0078		
p^{16}	1	0973^{11}	1535^{19}	2179^{18}
	1	9467^{21}	2356^{38}	4748^{36}
		-0031		
p^{32}	1	9436^{21}	2356^{38}	4748^{36}
p^{64}	1	8904^{43}	5551^{76}	2254^{73}

Examining the coefficients of p^{64}, we see that if the coefficients of p^{128} were formed they would be (to four significant digits) the squares of the coefficients of p^{64}, so we stop the process here and compute the roots of the equation as in Ex. 1, using, however, 4-place tables. We have

$$\log a^{64} = 43{\cdot}9496,$$

whence
$$a = \pm 4{\cdot}861.$$

In order to determine the sign of the root a, we note that $f(-4{\cdot}8) = +0{\cdot}368$ and $f(-4{\cdot}9) = -0{\cdot}259$, where $f(x) \equiv x^3 + 9x^2 + 23x + 14$. The equation has therefore a root between $-4{\cdot}8$ and $-4{\cdot}9$, and we have

$$a = -4{\cdot}861.$$

Also $\log a^{64}b^{64} = 76{\cdot}7444$; subtracting $\log a^{64}$, we find $\log |b| = \log 3{\cdot}254$, whence

$$b = -3{\cdot}254.$$

Again, $\log a^{64}b^{64}c^{64} = 73{\cdot}3530$; subtracting the value of $\log a^{64}b^{64}$, we find $\log |c| = \log 0{\cdot}8851$, whence

$$c = -0{\cdot}8851.$$

The required roots correctly to four significant figures are

$$-4{\cdot}861, \quad -3{\cdot}254, \quad -0{\cdot}8851.$$

Ex. 3.—*Apply the root-squaring method to approximate to the roots of the equation*

$$x^3+10x^2+31x+30=0$$

to three significant digits.

Ex. 4.—*Find correctly to seven digits the roots of the equation*

$$x^3+12x^2+44x+48=0.$$

55. Comparison of the Root-squaring Method with Bernoulli's Method.—The root-squaring method may be regarded as belonging to the same general type as that of Bernoulli, with which it may be connected in the following way.

Let the equation whose roots are to be determined be

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0. \tag{1}$$

Bernoulli's principle (§ 52) is that, if we select arbitrarily any values for $y(0)$, $y(1)$, . . ., $y(n-1)$, and then determine $y(n)$, $y(n+1)$, etc., in succession by the equation

$$y(t+n)+a_1y(t+n-1)+\ldots+a_ny(t)=0, \tag{2}$$

then the ratio of two successive members of the sequence of y's tends in general to a limit, which is the absolutely greatest root of the equation.

Now let s_p denote the sum of the pth powers of the roots of equation (1), and take $y(0)=s_0$, $y(1)=s_1$, . . ., $y(n-1)=s_{n-1}$. Since each root of equation (1) satisfies the equation

$$x^{t+n}+a_1x^{t+n-1}+\ldots+a_nx^t=0, \tag{3}$$

it is evident that if we form n equations by substituting in this equation the different roots in succession, and then add these n equations, we have at once

$$s_{t+n}+a_1s_{t+n-1}+\ldots+a_ns_t=0,$$

so that s_t, considered as a function of t, satisfies the difference equation (2). We see therefore that with the values we have chosen for the first n of the y's, $y(t+n)$ will be equal to s_{t+n}, and Bernoulli's formula yields the result that the absolutely greatest root of the equation (1) is

$$\operatorname*{Lt}_{k\to\infty}\frac{s_{k+1}}{s_k}$$

This is obviously closely related to the result of the root-squaring method, by which the root in question is given in the form $\operatorname*{Lt}_{k\to\infty}\sqrt[k]{s_k}$.*

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* The value of the numerically greatest root of an equation in the form $(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+\ldots)^{1/2n}$, where α, β, γ, . . . are the roots, and $\alpha>\beta>\gamma$, had been given as early as 1776 by Waring in his *Meditationes Analyticae*, p. 311. Euler used the method to find the roots of $J_0(x)$ in 1781.

56. **Application of the Root-squaring Method to determine the Complex Roots of an Equation.**—If the roots of an equation are not real the method of § 54 is equally applicable. Suppose, for example, that the equation is of the 5th degree, the roots in Encke's sense being a, $re^{i\phi}$, $re^{-i\phi}$, b, c, where a, b, c are real and $|a| > r > |b| > |c|$, so that the equation is

$$(x+a)(x+re^{i\phi})(x+re^{-i\phi})(x+b)(x+c)=0.$$

The equation whose Encke roots are the mth powers of these is

$$(x+a^m)(x+r^m e^{mi\phi})(x+r^m e^{-mi\phi})(x+b^m)(x+c^m)=0,$$

and if m is a large number, and we retain only the dominant part of the coefficient of each power of x, this reduces to

$$x^5 + a^m x^4 + 2a^m r^m \cos m\phi x^3 + a^m r^{2m} x^2 + a^m r^{2m} b^m x + a^m r^{2m} b^m c^m = 0. \quad (1)$$

The root a may be computed at once by the method already given in § 54. We now proceed to find the other roots.

It is evident from equation (1) that, corresponding to a pair of complex roots, there will be one coefficient, namely, $2a^m r^m \cos m\phi$, which fluctuates in sign when m takes in succession a set of increasing values (owing to the presence of the cosine factor). We also see that the value of r^2 corresponding to the pair of complex roots may be computed from the coefficients in the terms immediately preceding and immediately following the fluctuating term: in the above equation $r^2 = (a^m r^{2m}/a^m)^{1/m}$. Having found the value of r^2, we now find the values of the Encke roots b, c from the two last coefficients in equation (1). Let a', b', c' be the actual roots (*i.e.* roots with their proper sign) corresponding to the Encke roots a, b, c.

In order to compute the complex roots $re^{i\phi'}$, $re^{-i\phi'}$, we write them in the form $u+iv$ and $u-iv$, where $u = r\cos\phi'$, $v = r\sin\phi'$, so that $re^{i\phi'} + re^{-i\phi'} = 2u$. If the original equation is written

$$x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0,$$

we see at once that the sum of the roots satisfies the relation

$$-a_1 = a' + 2u + b' + c',$$

from which u may be found. Since u, r^2 are known, v may now be found from the relation $r^2 = u^2 + v^2$. The two complex roots are thus determined.

Ex.—1. *Solve the equation*

$$x^4 - 2x^3 - 6x^2 + 25x - 28 = 0.$$

	x^4.	x^3.	x^2.	x.	x^0.
	1	-2000	-6000	2500^1	-2800^1
	1	4000	3600^1	6250^2	7840^2
		12000	10000	-3360	
			-5600		
p^2	1	1600^1	8000^1	2890^2	7840^2
	1	2560^2	6400^3	8352^4	6147^5
		-1600	-9248	-12544	
			1568		
p^4	1	0960^2	-1280^3	-4192^4	6147^5
	1	9216^3	1638^6	1757^9	3779^{11}
		2560	8049	1574	
			1229		
p^8	1	1178^4	1092^7	3331^9	3779^{11}
	1	1388^8	1192^{14}	1110^{19}	1428^{23}
		-0218	-0785	-0825	
			0008		
p^{16}	1	1170^8	0415^{14}	0285^{19}	1428^{23}
	1	1369^{16}	1722^{27}	8123^{36}	2039^{46}
		-0008	-0667	-11852	
p^{32}	1	1361^{16}	1055^{27}	-3729^{36}	2039^{46}
	1	1852^{32}	1113^{54}	1391^{73}	4158^{92}
			0102	-4302	
p^{64}	1	1852^{32}	1215^{54}	-2911^{73}	4158^{92}
	1	3430^{64}	1476^{108}	8474^{146}	1729^{185}
			0011	-10104	
p^{128}	1	3430^{64}	1487^{108}	$[-2630^{146}]$	1729^{185}

It is evident that further equations are obtained by simply squaring the corresponding coefficients of the previous equation with the exception of the coefficient of x (which fluctuates in sign, thus indicating the presence of a pair of complex roots). The process of forming new equations may therefore be stopped here.

Taking logs, we obtain the results

$$\begin{aligned} \log a^{128} &= 64{\cdot}5353, \text{ whence } \log |a| = 0{\cdot}5042 \text{ and } a = 3{\cdot}193(-). \\ \log a^{128}b^{128} &= 108{\cdot}1724, \text{ whence } \log |b| = 0{\cdot}3409 \text{ and } b = 2{\cdot}193(+). \\ \log a^{128}b^{128}r^{256} &= 185{\cdot}2377, \text{ whence } \log r^2 = 0{\cdot}6021 \text{ and } r^2 = 4{\cdot}000. \end{aligned}$$

It may be remarked here that the most important numbers in determining the values of the quantities a, b, r^2 are the indices representing the powers of ten in the factor of the coefficients in the final equation. For example, in the equation denoted by p^{128} we obtain practically the same value of the roots a, b, if we neglect the last two digits of the numbers 3430 and 1487; but an error in the indices 64, 128 would give quite a different result. Since the above numbers cannot be relied upon as regards the accuracy of the fourth digit, we shall consider the final values of the roots to be correct only to three significant figures.

In order to compute the values of the complex roots $u \pm iv$, we see at once, from the coefficient of x^3 in the original equation, that

$$2u - 3{\cdot}193 + 2{\cdot}193 = 2,$$

whence $$u = 1{\cdot}500, \quad v = \pm \sqrt{(r^2 - u^2)} = \pm 1{\cdot}3228,$$

and the complex roots are $1{\cdot}500 \pm 1{\cdot}323i$.

Thus the required roots of the equation correctly to three significant digits are $-3{\cdot}19$, $2{\cdot}19$, $1{\cdot}50 \pm 1{\cdot}32i$. (As a matter of fact the above values of the roots are correct to four significant digits, the roots correctly to six places being $-3{\cdot}192582$, $2{\cdot}192582$, $1{\cdot}5 \pm 1{\cdot}322876i$.)

Ex. 2.—*Find the roots of the equation*

$$x^4 - 6x^3 + 11x^2 + 2x - 28 = 0.$$

57. Equations with more than one pair of Complex Roots.—The case where the given equation possesses two pairs of complex roots presents little additional difficulty. Suppose the process is applied to the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0, \tag{1}$$

and that the coefficients of two of the powers of x are found to fluctuate in sign, thus indicating the presence of two pairs of complex roots. Denote these complex roots by $re^{i\theta}$, $re^{-i\theta}$, $r'e^{i\phi}$, $r'e^{-i\phi}$. We shall assume that the values r^2, r'^2, together with the actual real roots a', b', c', have been computed as in the last section.

Writing $u = r\cos\theta$, $v = r\sin\theta$, $u' = r'\cos\phi$, $v' = r'\sin\phi$, the complex roots are denoted by $u \pm iv$, and $u' \pm iv'$, and their sum is $2(u+u')$. Writing equation (1) in the form

$$(x-a')(x-b')(x-c')\ .\ .\ .\ (x^2-2ux+r^2)(x^2-2u'x+r'^2)=0, \quad (2)$$

we see that the coefficient of x in this equation is a linear function of u, u' which we shall call $\phi(u, u')$.

Then
$$\phi(u, u') = a_{n-1}. \quad (3)$$

By comparing the coefficients of x^{n-1} in equations (1) and (2), we find

$$-a_1 = a' + b' + c' + .\ .\ . + 2u + 2u'. \quad (4)$$

The two last equations express the unknowns u, u' in terms of the known quantities $a', b', c', .\ .\ ., r^2, r'^2$ and may now be solved for u, u'; the values of v, v' being then determined from the equations $v^2 = r^2 - u^2$, $v'^2 = r'^2 - u'^2$. The two pairs of complex roots are thus known. It is evident that a similar method to that given here may be applied to solve equations with more than two pairs of complex roots.

Ex.—To solve the equation

$$x^7 + 7x^6 + 21x^5 + 63x^4 + 147x^3 + 189x^2 + 119x - 451 = 0.$$

As in Ex. 1, § 56, we first form the equations for p^2, p^4, p^8, . . . as follows:

	x^7.	x^6.	x^5.	x^4.	x^3.	x^2.	x.	
p^2	1	7000	-1470^2	0203^3	0911^4	-5610^4	1846^5	2034^5
p^4	1	3430^2	3699^4	1565^6	4865^7	-0133^9	5690^{10}	4137^{10}
p^8	1	0436^5	0391^9	-1355^{12}	6964^{15}	-5389^{18}	3249^{21}	1711^{21}
p^{16}	1	1119^9	2850^{17}	-4086^{24}	3644^{31}	-1621^{37}	1058^{43}	2928^{42}
p^{32}	1	0682^{18}	9044^{34}	-0411^{49}	1201^{63}	-5083^{74}	1119^{86}	8573^{84}
p^{64}	1	2842^{35}	8173^{69}	-2004^{98}	1438^{126}	-0104^{149}	1252^{172}	7350^{169}
p^{128}	1	6442^{70}	6680^{139}	1665^{196}	2068^{252}	-3590^{298}	1568^{344}	5402^{339}
p^{256}	1	4016^{141}	4462^{279}	$[0009^{392}]$	4277^{504}	$[0640^{597}]$	2459^{688}	2918^{679}

The coefficients of x^4 and x^2 fluctuate in sign, whence we infer the presence of two pairs of complex roots. Taking logs, we have

$256 \log a = 141{\cdot}6037$, whence $\log|a| = 0{\cdot}5531$ and $a' = -3{\cdot}574$.
$256 \log ab = 279{\cdot}6495$, $\log|b| = 0{\cdot}5392$ $b' = -3{\cdot}461$.
$256 \log abr^2 = 504{\cdot}6311$, $\log r^2 = 0{\cdot}8788$ $r^2 = 7{\cdot}565$.
$256 \log abr^2r'^2 = 688{\cdot}3908$, $\log r'^2 = 0{\cdot}7178$ $r'^2 = 5{\cdot}222$.
$256 \log abc^2r^2r'^2 = 679{\cdot}4651$, $\log|c| = \bar{1}{\cdot}9652$ $c' = 0{\cdot}923$.

In order to compute the values of u, u', we write down the value of the coefficient of x^6 in the original equation,

$$-7=-3{\cdot}574-3{\cdot}461+0{\cdot}923+2u+2u', \qquad (1)$$

and then the value of the coefficient of x,

$$119=r^2r'^2(a'b'+b'c'+c'a')+2a'b'c'(ur'^2+u'r^2). \qquad (2)$$

Substituting in these equations the known values of a', b', c', r^2, r'^2, we have

$$u=0{\cdot}681, \quad u'=-1{\cdot}125,$$

whence $$v=2{\cdot}665, \quad v'=1{\cdot}989.$$

The required roots of the given equation are

$$-3{\cdot}574, \quad -3{\cdot}461, \quad 0{\cdot}681\pm 2{\cdot}665i, \quad -1{\cdot}125\pm 1{\cdot}989i, \quad 0{\cdot}923,$$

in which values the last digit is uncertain.

[The roots correctly to four significant digits are $-3{\cdot}578$, $-3{\cdot}458$, $0{\cdot}684\pm 2{\cdot}664i$, $-1{\cdot}128\pm 1{\cdot}987i$, $0{\cdot}923$.]

58. The Solution of Equations with Coincident Roots by the Root-squaring Method.—When the root-squaring process is applied to an equation possessing coincident roots, it does not lead ultimately to an equation in which every coefficient is the square of the corresponding coefficient in the preceding equation. We shall now consider in what way the process may be applied to solve such an equation.

Suppose that the Encke roots of the given equation $f(x)=0$ are a, b, b', c, d, . . . where b, b' are coincident roots, and where $|a|>|b|>|c|>|d|$ . . . The equation, whose Encke roots are the mth powers of those of the given equation, will be denoted by

$$x^n+[a^m]x^{n-1}+[a^mb^m]x^{n-2}+[a^mb^mb'^m]x^{n-3}+\ldots=0, \qquad (1)$$

and, retaining only the dominant term in each coefficient, for large values of m this equation reduces to

$$x^n+a^mx^{n-1}+2a^mb^mx^{n-2}+a^mb^{2m}x^{n-3}+a^mb^{2m}c^mx^{n-4}+\ldots=0. \qquad (2)$$

We see that the coefficient of x^{n-2} does not follow the usual rule, viz. that when m is doubled the coefficient is approximately squared. Here, on the other hand, when m is doubled, the new coefficient is approximately *half* the square of the old one. This observation enables us to detect the presence of a repeated root. It is evident that the value of b may be computed in much the same way as r^2 in the case of complex roots, namely, by forming the quotient of the coefficients immediately following and immediately preceding the irregular coefficient. Thus from equation (2) we have

$$b = \sqrt[2m]{\{a^m b^{2m}/a^m\}},$$

the remaining roots, a, c, d, . . . being computed as before by the method of § 54.

Ex. 1.—*Solve by the root-squaring method the equation*

$$x^3 + 7x^2 + 16x + 12 = 0.$$

The successive equations are

$$\begin{array}{ll} p^2 & x^3 + 17x^2 + 88x + 144 = 0. \\ p^4 & x^3 + 113x^2 + 2848x + 20736 = 0. \\ p^8 & x^3 + 7073x^2 + 3{\cdot}425 \times 10^6 x + 4{\cdot}300 \times 10^8 = 0. \\ p^{16} & x^3 + 4{\cdot}318 \times 10^7 x^2 + 5{\cdot}648 \times 10^{12} x + 1{\cdot}849 \times 10^{17} = 0. \\ p^{32} & x^3 + 1{\cdot}853 \times 10^{15} x^2 + 1{\cdot}593 \times 10^{25} x + 3{\cdot}419 \times 10^{34} = 0. \end{array}$$

It is evident that the coefficients of the equation p^{64} would be the squares of the corresponding coefficients of p^{32} except in the case of the coefficient of x. We therefore denote the Encke roots of the equation by a, b, b', where b, b' are approximately equal, and where $|a| > |b|$. Taking logs, we have

$$\begin{aligned} \log a^{32} &= 15{\cdot}2679, \text{ whence } \log |a| = 0{\cdot}4771 \text{ and } |a| = 3{\cdot}000, \\ \log a^{32} b^{64} &= 34{\cdot}5339, \text{ whence } \log |b| = 0{\cdot}3010 \text{ and } |b| = 2{\cdot}000 \\ &\qquad = |b'|. \end{aligned}$$

The actual roots of the given equation are -3, -2, -2.

Ex. 2.—*Calculate by the root-squaring method the greatest root of the equation*

$$x^4 - 7x^3 + 7x^2 - 7x + 7 = 0.$$

Use the result to prove that the other three roots are nearly equal in modulus, but that only one of them is real.

59. Extension of the Root-squaring Method to the Roots of Functions given as Infinite Series.—The root-squaring process depends essentially on the fact that the coefficients of a polynomial (in which the coefficient of the highest power of x is unity) are the elementary symmetric functions of the roots; or, if the polynomial is divided throughout by the term independent of x, so that the term independent of x becomes unity, the coefficients are the elementary symmetric functions of the reciprocals of the roots. This, however, is true not merely for polynomials, but for all entire transcendental functions of genre zero,[*] and hence we can readily see that the root-squaring process is applicable to any entire transcendental function of genre zero; and, indeed, as has been shown by Polya,[†] the method may be used in connection with functions of any finite genre.

Ex. 1.—*To find by the root-squaring method the lowest roots of the Fourier-Bessel function.*

The Fourier-Bessel function may be defined by the power-series [‡]

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2.4^2} - \frac{z^6}{2^2.4^2.6^2} + \dots$$

$$= 1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \frac{x^4}{(4\,!)^2} + \frac{x^5}{(5\,!)^2} + \frac{x^6}{(6\,!)^2} + \frac{x^7}{(7\,!)^2} + \frac{x^8}{(8\,!)^2} + \dots$$

where $x = -\frac{1}{4}z^2$. Evaluating the terms on the right side of this equation, and retaining eleven places of decimals, we have

$$\begin{aligned} J_0(z) = 1 + x + 0{\cdot}25x^2 &+ 0{\cdot}027777{,}77778x^3 \\ &+ 0{\cdot}001736{,}11111x^4 \\ &+ 0{\cdot}000069{,}44444x^5 \\ &+ 0{\cdot}000001{,}92901x^6 \\ &+ 0{\cdot}000000{,}03937x^7 \\ &+ 0{\cdot}000000{,}00062x^8. \end{aligned}$$

The equation whose roots (in Encke's sense) are the squares of the roots of this equation is

$$0 = 1 + 0{\cdot}5x + 0{\cdot}010416{,}66666x^2 + 0{\cdot}000038{,}58024x^3 + 0{\cdot}000000{,}04306x^4 + \dots$$

Therefore the equation whose roots are the 4th powers of the roots of the original equation is

$$0 = 1 + 0{\cdot}229166{,}66667x + 0{\cdot}000070{,}01283x^2 + 0{\cdot}000000{,}00060x^3 + \dots$$

and the equation whose roots are the 8th powers of those roots is

$$0 = 1 + 0{\cdot}052377{,}33517x + 0{\cdot}000000{,}004625x^2.$$

Evidently we can take

$$x^8 = \frac{1}{0{\cdot}052377} = 19{\cdot}0923,$$

whence $x = -1{\cdot}4458$ and $z = 2\sqrt{-x} = \pm 2{\cdot}4048$.

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[*] On the subject of *genre* cf. Borel, *Leçons sur les fonctions entières* (Paris, 1900).

[†] *Zeitschrift für Math.* 63 (1915), p. 275.

[‡] Whittaker and Watson, *Modern Analysis*, § 17.1.

Therefore the lowest root of $J_0(z)$ is $z = 2{\cdot}4048$.

Ex. 2.—Calculate π from the property that $\pi/2$ is the lowest root of the series for $\cos x$, i.e.

$$0 = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \frac{x^8}{8\,!} - \dots$$

60. A Series Formula for the Root.—A method published in 1918[*] is somewhat different in character from those which have been described hitherto, since it furnishes a literal formula by mere substitution in which the root is obtained. The result may be stated thus:

The root of the equation

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \qquad (1)$$

which is the smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1} - \frac{a_0^2 a_2}{a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \frac{a_0^4 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \dots \qquad (2)$$

As a numerical example, consider the equation

$$x^3 - 4x^2 - 321x + 20 = 0.$$

Here $a_0 = 20$, $a_1 = -321$, $a_2 = -4$, $a_3 = 1$.

$$\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} = 103121, \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} = -33127121, \qquad \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix} = 337.$$

The smallest root of the equation is therefore

$$\frac{20}{321} - \frac{20^2 \times 4}{321 \times 103121} + \frac{20^3 \times 337}{103121 \times 33127121} + \dots$$

or $\qquad 0{\cdot}062305{,}30 - 0{\cdot}000048{,}36 + 0{\cdot}000000{,}79,$

or $\qquad 0{\cdot}062257{,}73$ correctly to seven decimal places.

The series converges rapidly when the ratio of the smallest root to every one of the other roots is small. In calculating any root of a given equation by the formula, it is therefore advisable in many cases first to transform the given equation by two or three root-squaring operations each of which replaces the equation operated on by an equation whose roots are the squares of its roots; or else, if an approximate value of the root is known, by Horner's process to reduce the roots of the given equation by a, where a is an approximate value of the required root, so that the required root of the new equation is small compared with any of the other roots.

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[*] Whittaker, *Proc. Edin. Math. Soc.* **36** (1918), p. 103. Cf. De Morgan, *Journ. Inst. Act.* **14** (1868), p. 353.

Proof of the Formula.—Let the roots of equation (1), supposed for the present to be of degree n, be $x_1, x_2, \ldots, x_n$. Then if z be any number whose modulus is smaller than each of the moduli of the roots, we have

$$\frac{a_0}{a_0+a_1z+a_2z^2+\ldots+a_nz^n}=\frac{1}{(1-z/x_1)(1-z/x_2)\ \ldots\ (1-z/x_n)}$$

$$=(1+z/x_1+z^2/x_1^2+\ldots)(1+z/x_2+z^2/x_2^2+\ldots)\ \ldots$$
$$(1+z/x_n+z^2/x_n^2+\ldots)$$
$$=1+P_1z+P_2z^2+P_3z^3+\ldots$$

where P_r denotes the sum of the homogeneous powers and products of the reciprocals of the roots taken r at a time.

Therefore

$$a_0=(a_0+a_1z+a_2z^2+\ldots)(1+P_1z+P_2z^2+P_3z^3+\ldots).$$

Equating coefficients of powers of z, we have

$$\begin{aligned}0&=a_1+a_0P_1,\\0&=a_2+a_1P_1+a_0P_2,\\0&=a_3+a_2P_1+a_1P_2+a_0P_3,\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}$$

whence *

$$P_1=-\frac{a_1}{a_0},\quad P_2=\frac{1}{a_0^2}\begin{vmatrix}a_1&a_0\\a_2&a_1\end{vmatrix},\quad P_3=-\frac{1}{a_0^3}\begin{vmatrix}a_1&a_0&0\\a_2&a_1&a_0\\a_3&a_2&a_1\end{vmatrix},\ \text{etc.}\qquad(3)$$

Now since $\begin{vmatrix}a_1&a_2\\a_0&a_1\end{vmatrix}=a_1^2-a_0a_2$, we see that the first two terms of the series (2) are equivalent to the single term

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* The formulae of equation (3) were known to Wronski, *Introd. à la philos. des math.* (1811), Paris.

$$-\frac{a_0 a_1}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} \text{ or } \frac{P_1}{P_2}. \tag{4}$$

Moreover, by Jacobi's theorem on the minors of the adjugate we have

$$a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}^2 - a_0^2 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}$$

and this shows that the term (4), together with the third term of the series (2), is equal to

$$-a_0 \frac{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} \text{ or } \frac{P_2}{P_3}. \tag{5}$$

Again, by Jacobi's theorem we have

$$\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}^2 - a_0^3 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}$$

and this shows that the term (5), together with the fourth term of the series (2), is equal to

$$-a_0 \frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} \text{ or } \frac{P_3}{P_4},$$

which is therefore equal to the sum of the first four terms of the series (2). Proceeding in this way, we see that the sum of the first s terms of the series is equal to P_{s-1}/P_s.

If now for simplicity we consider the case when $n=2$, so that there are only two roots, x_1 and x_2, of which we shall suppose x_1 to have the smaller modulus, we have

$$\frac{P_{s-1}}{P_s}=\frac{\dfrac{1}{x_1^{s-1}}+\dfrac{1}{x_1^{s-2}x_2}+\dfrac{1}{x_1^{s-3}x_2^2}+\ldots+\dfrac{1}{x_2^{s-1}}}{\dfrac{1}{x_1^{s}}+\dfrac{1}{x_1^{s-1}x_2}+\dfrac{1}{x_1^{s-2}x_2^2}+\ldots+\dfrac{1}{x_2^{s}}}$$

$$=x_1\frac{1+\dfrac{x_1}{x_2}+\dfrac{x_1^2}{x_2^2}+\ldots+\dfrac{x_1^{s-1}}{x_2^{s-1}}}{1+\dfrac{x_1}{x_2}+\dfrac{x_1^2}{x_2^2}+\ldots+\dfrac{x_1^{s-1}}{x_2^{s-1}}+\dfrac{x_1^{s}}{x_2^{s}}},$$

and since $\left|\frac{x_1}{x_2}\right|<1$, this gives at once

$$\operatorname*{Lt}_{s\to\infty}\frac{P_{s-1}}{P_s}=x_1.$$

Similar reasoning leads to the same result when $n>2$.

Thus the sum of the first s terms of the series is equal to P_{s-1}/P_s. As s increases indefinitely this tends to the limit x_1, where x_1 is that root of equation (1), which has the smallest modulus. Thus the theorem is established.

Ex. 1.—*Find the root, which is smallest in absolute value, of the equation*

$$x^3+14x^2-53268x+10472=0.$$

Ex. 2.—*Find the numerically smallest root of the equation*

$$x^5-25x^4+200x^3-600x^2+1200x-500=0.$$

61. **General Remarks on the Different Methods.**—The process of calculating a root of an equation $f(x)=0$ may be regarded as consisting in general of three stages:

I. *Locating roughly the position of the root.*—This is, in general, best done by plotting a rough graph; in the case when $f(x)$ is a polynomial, it may also be done by divided differences, or by the rules of Descartes, Fourier, and Sturm, which are explained in works on the algebraical theory of equations.

II. *Transforming the equation so as to isolate this root from the other roots of the equations.*—Let x_1 be the required root of the equation $f(x)=0$, and let $x_2, x_3, \ldots$ be the other roots. Then, performing a transformation $x=\phi(y)$, where ϕ is some given function, we obtain an equation $F(y)=0$, which we shall suppose to have roots $y_1, y_2, y_3, \ldots$ corresponding respectively to the roots $x_1, x_2, x_3, \ldots$ This transformation is to be chosen so that the root y_1 bears to the other roots $y_2, y_3, \ldots$ ratios which are small in absolute value. The root y_1 is then said to be *isolated.* For this purpose we may use, for example, Horner's process, or the root-squaring process or inverse interpolation.

III. *Calculating the root of the transformed equation.*—This may be done rapidly by one of the iterative methods, *e.g.* Newton's (§ 44), or Bernoulli's (§ 52), or by Whittaker's series (§ 60). Having found y_1, the required root of the original equation is given by the formula $x_1 = \phi(y_1)$.

62. **The Numerical Solution of the Cubic.**—For the solution of cubic and quartic equations some special methods are available which cannot be applied to equations of higher degree. To one of these, known as the *trisection method* of solving the cubic, we shall briefly now refer. The term in x^2 may be supposed to have been removed in the usual way, so that the cubic may be written in the form

$$x^3 - qx - r = 0.$$

There are two principal cases to consider:

(i.) If $27r^2 > 4q^3$, the cubic has one real root and two complex roots, which may be found by Cardan's formula

$$x = \{\tfrac{1}{2}r + (\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)^{\frac{1}{2}}\}^{\frac{1}{3}} + \{\tfrac{1}{2}r - (\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}, \qquad (1)$$

or else (if q and r are both positive) by finding ϕ such that

$$\cosh \phi = \frac{3^{\frac{3}{2}}}{2} \cdot \frac{r}{q^{\frac{3}{2}}},$$

when the real root is given by formula (1)

$$x = \frac{2}{\sqrt{3}} q^{\frac{1}{2}} \cosh \tfrac{1}{3}\phi, \qquad (2)$$

or (if q is negative and r positive) by finding ϕ such that

$$\sinh \phi = \frac{3^{\frac{3}{2}}}{2} \cdot \frac{r}{(-q)^{\frac{3}{2}}},$$

when the real root is given by the formula

$$x = \frac{2}{\sqrt{3}} (-q)^{\frac{1}{2}} \sinh \tfrac{1}{3}\phi. \qquad (3)$$

We can always suppose that r is positive, since changing the sign of r merely changes the signs of the roots.

(ii.) If $27r^2 < 4q^3$, the cubic has all its roots real. In this case we find the smallest positive angle ϕ such that

$$\cos\phi = \frac{3^{\frac{3}{2}}}{2} \cdot \frac{r}{q^{\frac{3}{2}}},$$

when the real roots are given by

$$\left.\begin{aligned} x_1 &= \frac{2}{\sqrt{3}} q^{\frac{1}{2}} \cos\frac{\phi}{3} \\ x_2 &= -\frac{2}{\sqrt{3}} q^{\frac{1}{2}} \cos\frac{\pi-\phi}{3} \\ x_3 &= -\frac{2}{\sqrt{3}} q^{\frac{1}{2}} \cos\frac{\pi+\phi}{3} \end{aligned}\right\}. \tag{4}$$

Ex. 1.—*Consider the equation*

$$x^3 + 9x^2 + 23x + 14 = 0.$$

Writing $x = y - 3$ to remove the second term, the equation becomes

$$y^3 - 4y - 1 = 0. \tag{1}$$

There are three real roots since $4.4^3 > 27.1^2$.

We have at once

$$\cos\phi = \tfrac{1}{16} \cdot 3^{\frac{3}{2}} \text{ and } \log\cos\phi = \tfrac{3}{2}\log 3 - \log 16 = \log\cos 71^\circ\, 2'\, 56''{\cdot}4,$$

so that
$$\tfrac{1}{3}\phi = 23^\circ\, 40'\, 58''{\cdot}8.$$

One value of y is now obtained by writing

$$\begin{aligned} \log y_1 &= \log 4 - \tfrac{1}{2}\log 3 + \log\cos 23^\circ\, 40'\, 58''\, 8 \\ &= 0{\cdot}3252913 \\ &= \log 2{\cdot}114907, \end{aligned}$$

so
$$y_1 = 2{\cdot}114907. \tag{2}$$

For the second root we have

$$\begin{aligned} \log(-y_2) &= \log 4 - \tfrac{1}{2}\log 3 + \log\cos(60^\circ - 23^\circ\, 40'\, 58''{\cdot}8) \\ &= 0{\cdot}2697010 \\ &= \log 1{\cdot}860806, \end{aligned}$$

so
$$y_2 = -1{\cdot}860806. \tag{3}$$

Lastly,

$$\begin{aligned} \log(-y_3) &= \log 4 - \tfrac{1}{2}\log 3 + \log\cos(60^\circ + 23^\circ\, 40'\, 58''{\cdot}8) \\ &= 9{\cdot}4050073 \\ &= \log 0{\cdot}2541015, \end{aligned}$$

so
$$y_3 = -0{\cdot}2541015. \tag{4}$$

The required roots are therefore $-0{\cdot}885093$, $-4{\cdot}860806$, $-3{\cdot}2541015$.

Ex. 2.—*Solve the equation*

$$x^3 + 3x^2 + 5x - 27 = 0$$

by the trisection method.

63. Graphical Method of solving Equations.—We shall now consider the solution of an equation $f(x) = 0$ by a graphical procedure. We first write the equation in the form

$$f_1(x) = f_2(x)$$

(as may usually be done in several ways), and draw graphs of the equations $y = f_1(x)$, $y = f_2(x)$. For example, to solve the equation $x^3 - 4x + 6 = 0$, the curves might be taken to be $y_1 = x^3$, $y_2 = 4x - 6$. The abscissae of the points of intersection of these curves evidently satisfy the equation $f(x) = 0$, and are thus the roots of the given equation.

Let one of these abscissae, as measured on the graph, be denoted by a_1; we must, of course, recognise that a_1 is only an approximation to the root, since the accuracy of the reading is limited by the usual defects of graphical representation. To overcome this difficulty we now repeat the drawing of the portion of the graph near the abscissa $x = a_1$ on a larger scale. From the new diagram we find a new value $x = a_2$ such that a_2 is a closer approximation to the root than a_1. Proceeding in this way, we form a sequence of values $a_1, a_2, a_3, \ldots$, which approach continually closer to the correct value of the root of the equation.

Ex.—*Find the real root of the equation*

$$\cos x = x$$

correctly to seven places of decimals, having given

$$\begin{aligned} \cos 42^\circ\, 20'\, 40'' &= 0{\cdot}739108{,}82, \\ \cos 42^\circ\, 20'\, 50'' &= 0{\cdot}739076{,}16, \\ \cos 42^\circ\, 21'\, 0'' &= 0{\cdot}739043{,}50, \end{aligned}$$

and

$$\pi/648000 = 0{\cdot}000004{,}848137.$$

Suppose $x = y''$, then the given equation becomes

$$0{\cdot}000004{,}848137y = \cos y. \qquad (1)$$

We first plot the curves

$$\left.\begin{aligned} f_1(y) &= 0{\cdot}000004{,}848137y \\ f_2(y) &= \cos y \end{aligned}\right\}, \qquad (2)$$

using the following data :

$$42^\circ\ 20'\ 40'' = 152440'', \quad 0{\cdot}000004{,}848137 \times 152440 = 0{\cdot}739050{,}0,$$
$$42^\circ\ 20'\ 50'' = 152450'', \quad 0{\cdot}000004{,}848137 \times 152450 = 0{\cdot}739098{,}5.$$

Fig. 12 suggests as a solution the value $y = 42^\circ\ 20'\ 47''{\cdot}3$, so the first approximation for y is

$$a_1 = 0{\cdot}739084{,}98.$$

Now
$$\cos 42^\circ\ 20'\ 47''{\cdot}2 = 0{\cdot}739085{,}30,$$
$$\cos 42^\circ\ 20'\ 47''{\cdot}3 = 0{\cdot}739084{,}98,$$

and
$$0{\cdot}000004{,}848137 \times 152447{\cdot}2 = 0{\cdot}739084{,}91,$$
$$0{\cdot}000004{,}848137 \times 152447{\cdot}3 = 0{\cdot}739085{,}40.$$

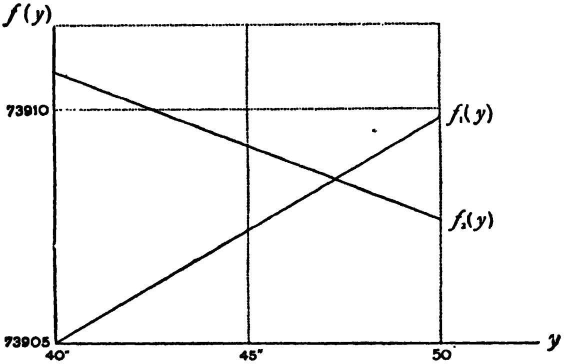

FIG. 12.

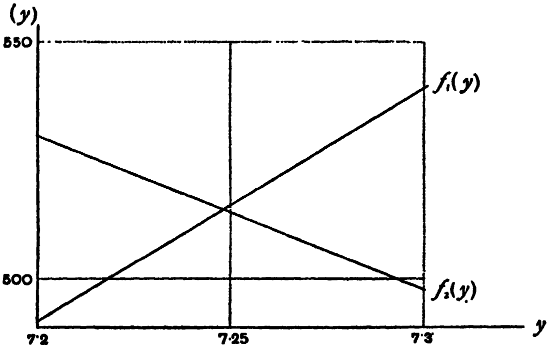

FIG. 13.

Fig. 13 gives as a solution $y = 42^\circ\ 20'\ 47''{\cdot}248$, and the value of the cosine is then 0·739085,14. The second approximation to the root is therefore

$$a_2 = 0{\cdot}739085{,}14,$$

which is the required value.

64. Nomography.—Nomography is a special kind of graphical calculation which differs from other graphical methods in this respect: that with graphical methods generally it is necessary to draw a fresh diagram for every problem that is to be solved, whereas in nomography it is possible to use the same diagram over and over again for the solution of different problems, so long as these problems belong to the same type, and differ only in regard to the numerical data occurring in them. For example, there are many well-known graphical solutions of the quadratic equation in which the roots are obtained as the intersections of a circle with a straight line. The practical objection to these solutions is that a fresh diagram has to be drawn for each particular quadratic equation that has to be solved; and the time occupied in constructing the diagram is greater than the time required to solve the equation by the ordinary arithmetical method. This objection no longer applies if the diagram is of such a nature that when once constructed it can be used for *any* quadratic, whatever be the values of the coefficients. Such a diagram, in which the construction is made once for all and is applicable to any number of special cases, is called a *nomogram.*

The detailed treatment of the subject, which is an extensive one, is somewhat beyond the scope of the present book; the reader may be referred to the works of M. d'Ocagne* and S. Brodetsky†; we shall confine ourselves to illustrating the fundamental idea of nomography by describing a simple nomogram,‡ by means of which quadratic equations may be solved at sight.

The accompanying diagram consists of two rectangular axes and a circle touching one of them (which we may regard as horizontal) at the origin. The circle may be of any arbitrary radius R. The horizontal axis is graduated so that the graduation p on it is at a distance $2R/p$ from the origin, and the other axis is graduated so that the graduation p on it is at a distance $2R/(1-p)$ from the origin. The graduation at any point of the circle is the same (with sign reversed) as the graduation at that point of the horizontal axis which is derived from it by projection from the highest point of the circle.

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* *Traité de nomographie* (1899) and *Calcul graphique et nomographie* (1914).

† *A First Course in Nomography* (1920).

‡ E. T. Whittaker, *Edin. Math. Soc. Notes*, No. 19 (Dec. 1915), p. 215.

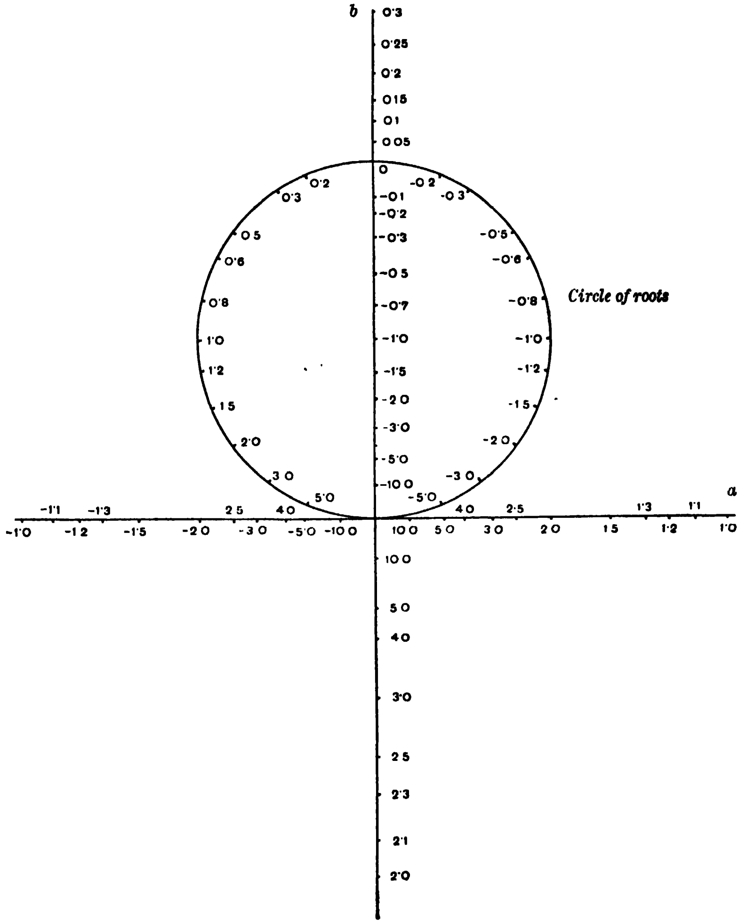

FIG. 14.—Nomogram for the Solution of $x^2+ax+b=0$.

The method of using the nomogram is as follows: *Let the equation to be solved be*

$$x^2 + ax + b = 0.$$

Find the point on the horizontal axis at which the reading is a, and the point on the vertical axis at which the reading is b.

Imagine these two points joined by a straight line (*e.g.* by stretching a thread between them, or by laying a straight-edge across). *Where the line meets the circle, read off the graduations on the circle: these are the required roots of the quadratic.*

The proof may be left to the reader.

If the line is not conveniently situated on the diagram, we may replace the given equation, *e.g.* by the equation whose roots are the roots of the given equation with signs reversed, or the equation whose roots are the roots of the given equation multiplied or divided by 10 or some power of 10.

MISCELLANEOUS EXAMPLES ON CHAPTER VI

1. Find the positive root of the equation

$$x + x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} = 5.$$

2. Find the positive root of each of the following equations

$$(\alpha) \quad x \log_e x = 18, \qquad (\beta) \quad \log_e (x+1)/(x-1) = 2x.$$

3. Find correctly to four significant digits the positive root of the equation

$$x^x + 5x = 1000,$$

using either the Newton-Raphson method or the method of iteration.

4. Compute to seven places the positive root of

$$x^7 + 28x^4 = 480.$$

5. Locate roughly the real roots of the equation

$$x^4 - 3x^2 + 75x - 10000 = 0,$$

and calculate one of them by the Newton-Raphson method to four places of decimals.

6. Show by the Newton-Raphson method that a root of the equation

$$y^3 + a(x+a)y - x^3 - 2a^3 = 0$$

is
$$y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} + \cdots$$

(Wallis, *Algebra* (1685).)

7. Solve the equation

$$x^4 + 2x^3 - x^2 - x - 631064798 = 0$$

by Horner's method.

(De Morgan.)

8. Apply the root-squaring method to approximate to the roots of the following equations to three significant figures:

$$(\alpha)\ 2x^3 - x^2 - 7x + 5 = 0, \qquad (\beta)\ x^3 + x^2 - 5x + 2 = 0.$$

9. Find by the root-squaring method the numerically greatest root of the equation

$$4x^4 + 128x^3 - 95x^2 + 24x - 2 = 0,$$

and determine the nature of the remaining roots.

10. Show that the root-squaring method may be applied to determine the coincident roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 0.$$

11. Determine z from the equation

$$z = e \sin (n + z),$$

where e is the eccentricity of a planet's orbit,
n is the mean anomaly,
having given the values $e = 0{\cdot}208$, $n = 30° = 0{\cdot}5235988$.

ADDITIONAL REFERENCES

A. Ostrowski, " Recherches sur la méthode de Graeffe ", *Acta Math.* **72** (1940), pp. 99, 157.

L. Collatz, " Das Hornersche Schema bei complazen Wurzeln ", *Z. Angew. Math.* **20** (1940), p. 235.

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S. N. Lin, " A method of successive approximations for evaluating the real and complex roots of cubic and higher-order equations ", *J. Math. Phys.* **20** (1941), p. 231.

CHAPTER VII

NUMERICAL INTEGRATION AND SUMMATION

65. Introduction.—In the present chapter we shall show how to calculate the numerical value of a definite integral

$$\int_a^b f(x)dx$$

when $f(x)$ is a function whose numerical value is known for values of x between a and b.

In works on the Integral Calculus it is shown that the above definite integral can be found, provided we can first find the *indefinite* integral of $f(x)$, say $\mathrm{F}(x)$: for the value of the definite integral is then $\mathrm{F}(b) - \mathrm{F}(a)$. This method is, however, of very limited application: for even when $f(x)$ is given as a compound of the well-known elementary functions x^n, e^x, $\log x$, its indefinite integral cannot in general be expressed as a compound of a finite number of these functions: and when $f(x)$ is not given in terms of the known elementary functions, but is merely specified by a table of numerical values, the indefinite-integral method is altogether inapplicable. The methods described in the present chapter, which are of general application, are therefore of great practical importance.

The problem of calculating the sum of a sequence

$$u_1 + u_2 + u_3 + u_4 + . \ . \ . + u_n$$

is closely connected with the problem of numerical integration, and is therefore included in the discussion.

66. The Approximate Value of a Definite Integral.—A definite integral

$$\int_a^{a+rw} f(x)dx$$

may be regarded as the measure of the area included between the curve $y=f(x)$, the ordinates $x=a$ and $x=a+rw$, and the axis of x.

Let P and Q be the two points on the curve whose abscissae are a and $a+rw$ respectively, so that the area in question is PLMQ in the diagram.

Let the base LM be divided into r parts each of length w, at the points U, V, W, . . ., and let the corresponding points on the curve be H, J, K, . . . Then we have

area of quadrilateral PLUH $= \tfrac{1}{2}w\{f(a)+f(a+w)\}$,

" " HUVJ $= \tfrac{1}{2}w\{f(a+w)+f(a+2w)\}$,

etc.,

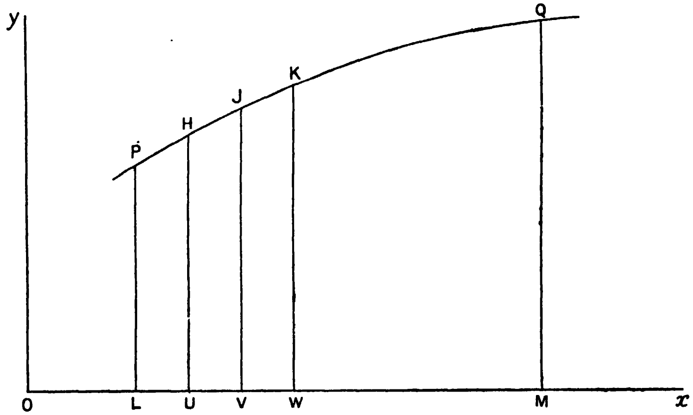

FIG. 15.

and the sum of the areas of the quadrilaterals PLUH, HUVJ, . . .

$$= w\{\tfrac{1}{2}f(a)+f(a+w)+f(a+2w)+\ldots+f(a+\overline{r-1}\,.\,w)+\tfrac{1}{2}f(a+rw)\}.$$

This sum may be regarded as a rough approximation to the area PLMQ, so that we have

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \tfrac{1}{2}f(a)+f(a+w)+f(a+2w)+\ldots$$
$$+f(a+\overline{r-1}\,.\,w)+\tfrac{1}{2}f(a+rw)+\mathrm{T},$$

where T denotes certain correction-terms.

We shall now show how these correction-terms may be found.

67. The Euler-Maclaurin Formula.—Let us consider the case in which the function $f(x)$ is e^{vx} where v is independent of x. Then the above formula becomes

$$\frac{1}{w}\int_a^{a+rw} e^{vx}dx = \tfrac{1}{2}e^{va} + e^{v(a+w)} + e^{v(a+2w)} + \ldots + e^{va+(r-1)vw} + \tfrac{1}{2}e^{va+vrw} + \text{T}, \quad (1)$$

where T denotes the correction-terms; or (performing the integration)

$$\frac{e^{va}(e^{vrw}-1)}{vw} = e^{va}\{1 + e^{vw} + e^{2vw} + \ldots + e^{(r-1)vw}\} + \tfrac{1}{2}e^{va}(e^{vrw}-1) + \text{T}$$

$$= \frac{e^{va}(e^{vrw}-1)}{e^{vw}-1} + \tfrac{1}{2}e^{va}(e^{vrw}-1) + \text{T},$$

so

$$\frac{1}{vw} = \frac{1}{e^{vw}-1} + \tfrac{1}{2} + \frac{\text{T}}{e^{va}(e^{vrw}-1)}. \quad (2)$$

Now we have

$$\frac{1}{e^{\theta}-1} = \frac{1}{\theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots} = \frac{1}{\theta}\left(1 + \frac{\theta}{2} + \frac{\theta^2}{6} + \ldots\right)^{-1}$$

$$= \frac{1}{\theta}\left(1 - \frac{1}{2}\theta + \frac{1}{12}\theta^2 + \text{terms in } \theta^4, \text{ etc.}\right)$$

$$= \frac{1}{\theta} - \frac{1}{2} + \frac{1}{12}\theta - \frac{1}{720}\theta^3 + \ldots$$

It is customary to write this expansion

$$\frac{1}{e^{\theta}-1} = \frac{1}{\theta} - \frac{1}{2} + \frac{\text{B}_1}{2!}\theta - \frac{\text{B}_2}{4!}\theta^3 + \frac{\text{B}_3}{6!}\theta^5 - \ldots, \quad (3)$$

where the numbers B, which are called the *Bernoullian numbers*, have the numerical values

$$\text{B}_1 = \frac{1}{6}, \quad \text{B}_2 = \frac{1}{30}, \quad \text{B}_3 = \frac{1}{42}, \quad \text{B}_4 = \frac{1}{30}, \quad \text{B}_5 = \frac{5}{66}, \quad \text{B}_6 = \frac{691}{2730} \ldots$$

Comparing equations (2) and (3) we have

$$\text{T} = e^{va}(e^{vrw}-1)\left(-\frac{\text{B}_1}{2!}vw + \frac{\text{B}_2}{4!}v^3w^3 - \frac{\text{B}_3}{6!}v^5w^5 + \ldots\right).$$

Now if $f(x) = e^{vx}$, we have $f'(x) = ve^{vx}$, $f''(x) = v^2e^{vx}$, . . . and therefore when $f(x) = e^{vx}$, we have

$$T = -\frac{B_1 w}{2!}\{f'(a+rw)-f'(a)\}+\frac{B_2 w^3}{4!}\{f'''(a+rw)-f'''(a)\}$$
$$-\frac{B_3 w^5}{6!}\{f^{\text{v}}(a+rw)-f^{\text{v}}(a)\}+\ldots,$$

so that in the case where $f(x)=e^{vx}$, we have the formula

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \tfrac{1}{2}f(a)+f(a+w)+f(a+2w)+\ldots$$
$$+f(a+\overline{r-1}\,.w)+\tfrac{1}{2}f(a+rw)$$
$$-\frac{B_1 w}{2!}\{f'(a+rw)-f'(a)\}+\frac{B_2 w^3}{4!}\{f'''(a+rw)-f'''(a)\}$$
$$-\frac{B_3 w^5}{6!}\{f^{\text{v}}(a+rw)-f^{\text{v}}(a)\}+\ldots \qquad (4)$$

or, as we may more briefly write it,

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \left(\frac{1}{2}f_0+f_1+f_2+\ldots+f_{r-1}+\frac{1}{2}f_r\right)$$
$$-\frac{w}{12}(f_r'-f_0')+\frac{w^3}{720}(f_r'''-f_0''')-\frac{w^5}{30240}(f_r^{\text{v}}-f_0^{\text{v}})+\ldots \qquad (5)$$

Now let $f(x)$ be an arbitrary function of x. We shall assume that it is possible to represent $f(x)$ between the finite bounds $x=a$ and $x=a+rw$ to a sufficient degree of approximation by means of a sum of terms of the form Ae^{vx}, where A and v are independent of x: say let

$$f(x)=A_1e^{v_1x}+A_2e^{v_2x}+A_3e^{v_3x}+\ldots,$$

where A_1, A_2, A_3, . . ., v_1, v_2, v_3, . . . are suitably chosen constants.

Applying formula (4) to each term of this sum, and adding the results, we see that the formula (4) or (5) may be applied to such an arbitrary function $f(x)$ in general. It is known as the *Euler-Maclaurin formula*, having been discovered independently by Euler and by Maclaurin in the years 1730–1740.*

Ex.—To compute $\int_{100}^{105}\frac{dx}{x}$ *correctly to eight places of decimals.*

In the Euler-Maclaurin formula put $a=100$, $w=1$, $r=5$, $f(x)=\frac{1}{x}$.

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* In both cases the publication was several years subsequent to the discovery. Cf. Euler, *Comm. Acad. Sci. Imp. Petrop.* 6 (1738), p. 68, and Maclaurin, *Treatise of Fluxions* (1742), p. 672.

Thus

$$\int_{100}^{105}\frac{dx}{x}=\frac{1}{2}\,\frac{1}{100}+\frac{1}{101}+\frac{1}{102}+\frac{1}{103}+\frac{1}{104}+\frac{1}{2}\,\frac{1}{105}$$
$$-\frac{1}{12}\left(-\frac{1}{105^2}+\frac{1}{100^2}\right)+\frac{1}{120}\left(-\frac{1}{105^4}+\frac{1}{100^4}\right)+\ldots$$

$$\begin{array}{ll}
=0{\cdot}005000\,000 & -\dfrac{1}{12}\left(\begin{array}{l}0{\cdot}000100\,000\\ -0{\cdot}000090\,703\end{array}\right)\\
0{\cdot}009900\,990 & \\
0{\cdot}009803\,922 & \\
0{\cdot}009708\,738 & \\
0{\cdot}009615\,385 & \\
0{\cdot}004761\,905 & \\
 & \\
=0{\cdot}048790\,940 & -0{\cdot}000000\,775\\
=0{\cdot}048790\,165 &
\end{array}$$

The calculated value of the integral is therefore 0·048790,165.

This result may be verified by computing $\log_e(105/100)$ from the logarithmic series, as follows :

$$\log_e\frac{105}{100}=\frac{5}{100}-\frac{1}{2}\,\frac{5^2}{100^2}+\frac{1}{3}\,\frac{5^3}{100^3}-\frac{1}{4}\,\frac{5^4}{100^4}+\frac{1}{5}\,\frac{5^5}{100^5}-\frac{1}{6}\,\frac{5^6}{100^6}+\ldots$$

$$\begin{array}{lrr}
= & 0{\cdot}050000\,000 & -0{\cdot}001250\,000\\
 & +0{\cdot}000041\,667 & -0{\cdot}000001\,562\\
 & +0{\cdot}000000\,062 & -0{\cdot}000000\,003\\
= & 0{\cdot}050041\,729 & -0{\cdot}001251\,565\\
= & 0{\cdot}048790{,}164 &
\end{array}$$

which agrees with the previous result to one unit in the ninth place of decimals.

68. **Application to the Summation of Series.**—The Euler-Maclaurin formula is often used in the computation of sums, when the integral which occurs in it can be calculated by other methods; the method will be obvious from the following examples:

Ex. 1.—If it is required *to obtain correctly to nine places of decimals the value of*

$$\frac{1}{201^2}+\frac{1}{203^2}+\frac{1}{205^2}+\ldots+\frac{1}{299^2},$$

we have from the Euler-Maclaurin formula with $a=201$, $w=2$, $r=50$, $f(x)=1/x^2$,

$$\frac{1}{2}\int_{201}^{301}\frac{dx}{x^2}=\frac{1}{2}\,\frac{1}{201^2}+\frac{1}{203^2}+\ldots+\frac{1}{299^2}+\frac{1}{2}\,\frac{1}{301^2}$$
$$-\frac{1}{6}\left(-\frac{2}{301^3}+\frac{2}{201^3}\right)+\ldots$$

so

$$\frac{1}{201^2}+\ldots+\frac{1}{299^2}=\frac{1}{2}\left(\frac{1}{201}-\frac{1}{301}\right)+\frac{1}{2}\left(\frac{1}{201^2}-\frac{1}{301^2}\right)$$
$$+\frac{1}{3}\left(\frac{1}{201^3}-\frac{1}{301^3}\right)+\ldots$$

$$\begin{aligned}&=0{\cdot}000826\ 4325\\ &\ 0{\cdot}000006\ 8575\\ &\ 0{\cdot}000000\ 0288\end{aligned}$$

$=0{\cdot}000833\ 319$, which is the required value.

Ex. 2.—*To compute the sum of the infinite series*

$$\frac{1}{101^2}+\frac{1}{103^2}+\frac{1}{105^2}+\frac{1}{107^2}+\ldots$$

If in the Euler-Maclaurin formula we put $f(x)=1/x^2$, $w=2$, $r=\infty$, we have

$$\frac{1}{2a}=\frac{1}{2a^2}+\frac{1}{(a+2)^2}+\frac{1}{(a+4)^2}+\frac{1}{(a+6)^2}+\ldots-\frac{1}{3a^3}+\frac{4}{15a^5}-\ldots$$

so that

$$\frac{1}{a^2}+\frac{1}{(a+2)^2}+\frac{1}{(a+4)^2}+\ldots=\frac{1}{2a}+\frac{1}{2a^2}+\frac{1}{3a^3}-\frac{4}{15a^5}+\ldots$$

Putting $a=101$, this gives

$$\frac{1}{101^2}+\frac{1}{103^2}+\frac{1}{105^2}+\ldots=\frac{1}{2.101}+\frac{1}{2.101^2}+\frac{1}{3.101^3}-\frac{4}{15.101^5}+\ldots$$

$$\begin{aligned}&=0{\cdot}004950{,}49505 \quad -0{\cdot}000000{,}00003\\ &+0{\cdot}000049{,}01480\\ &+0{\cdot}000000{,}32353\end{aligned}$$

$=0{\cdot}004999{,}83335$, which is the required sum.

Ex. 3.—*Calculate correctly to ten places the sum of the infinite series*

$$\frac{1}{11^3}+\frac{1}{12^3}+\frac{1}{13^3}+\frac{1}{14^3}+\ldots$$

69. The Sums of Powers of the Whole Numbers.—The sums of the first, second, and third powers of the first n whole numbers are, as is well known,

$$\begin{aligned}1+2+3+\ldots+n&=\tfrac{1}{2}n^2+\tfrac{1}{2}n,\\ 1^2+2^2+3^2+\ldots+n^2&=\tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n,\\ 1^3+2^3+3^3+\ldots+n^3&=\tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2.\end{aligned}$$

We shall now show that the Euler-Maclaurin formula enables us to find readily the sum of the pth powers of the first r whole numbers, where p is any positive whole number.

For taking $a=0$, $w=1$, $f(x)=x^p$ in the formula

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \frac{1}{2}f_r\right) - \frac{w}{12}(f_r' - f_0') + \frac{w^3}{720}(f_r''' - f_0''') - \frac{w^5}{30240}(f_r^{\text{v}} - f_0^{\text{v}}) + \ldots$$

we have

$$\frac{r^{p+1}}{p+1} = 1^p + 2^p + 3^p + \ldots + (r-1)^p + \frac{1}{2}r^p - \frac{1}{12}pr^{p-1} + \frac{1}{720}p(p-1)(p-2)r^{p-3} - \frac{1}{30240}p(p-1)(p-2)(p-3)(p-4)r^{p-5} + \ldots,$$

and therefore

$$1^p + 2^p + 3^p + \ldots + r^p = \frac{r^{p+1}}{p+1} + \frac{r^p}{2} + \frac{p}{12}r^{p-1} - \frac{p(p-1)(p-2)}{720}r^{p-3} + \frac{p(p-1)(p-2)(p-3)(p-4)}{30240}r^{p-5} - \ldots,$$

the last term being that in r or r^2.

This is the general formula for a sum of positive powers: it was first published in James Bernoulli's posthumous work, *Ars Conjectandi*, in 1713.*

Ex.—Show that the sum of the 7th and 5th powers of the first n whole numbers is double the square of the sum of their cubes.

We have at once

$$1^7 + 2^7 + 3^7 + \ldots + n^7 = \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2.$$

Similarly

$$1^5 + 2^5 + 3^5 + \ldots + n^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2.$$

We have therefore only to prove that

$$(\tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2) + (\tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2) = \tfrac{1}{8}n^4(n+1)^4,$$

which is evidently true.

70. Stirling's Approximation to the Factorial.—In the Euler-Maclaurin expansion

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \frac{1}{2}f_r\right) - \frac{w}{12}(f_r' - f_0') + \frac{w^3}{720}(f_r''' - f_0''') - \ldots$$

---

* *Ars Conj.* p. 97.

write $w=1, f(x)=\log x$. The formula then gives

$$\int_a^{a+r}\log x\,dx=\frac{1}{2}\log a+\log(a+1)+\ldots+\log(a+r-1)$$

$$+\frac{1}{2}\log(a+r)-\frac{1}{12}\left(\frac{1}{a+r}-\frac{1}{a}\right)+\frac{1}{360}\left\{\frac{1}{(a+r)^3}-\frac{1}{a^3}\right\}-\ldots$$

or

$$\left(a+r+\frac{1}{2}\right)\log(a+r)-\left(a+\frac{1}{2}\right)\log a-r$$

$$=\log\{(a+1)(a+2)\ \ldots\ (a+r)\}-\frac{1}{12}\left(\frac{1}{a+r}-\frac{1}{a}\right)$$

$$+\frac{1}{360}\left(\frac{1}{(a+r)^3}-\frac{1}{a^3}\right)-\ldots$$

Writing n for $(a+r)$, this becomes

$$\left(n+\frac{1}{2}\right)\log n-\left(a+\frac{1}{2}\right)\log a-n+a=\log(n!)-\log(a!)$$

$$-\frac{1}{12}\left(\frac{1}{n}-\frac{1}{a}\right)+\frac{1}{360}\left(\frac{1}{n^3}-\frac{1}{a^3}\right)-\ldots$$

We have therefore

$$\log(n!)=\mathrm{C}+\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots \qquad (1)$$

where C is independent of n.

In order to determine C, we refer to Wallis' formula

$$\frac{\pi}{2}=\operatorname*{Lt}_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\ \cdots\ \frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right)$$

or
$$\frac{\pi}{2}=\operatorname*{Lt}_{n\to\infty}\frac{2^{4n}(n!)^4}{\{(2n)!\}^2(2n+1)},$$

and therefore

$$\log\frac{\pi}{2}=\operatorname*{Lt}_{n\to\infty}[4n\log 2+4\log n!-2\log(2n!)-\log(2n+1)].$$

Substituting for the logs of the factorials from (1), we have

$$\log\frac{\pi}{2}=\operatorname*{Lt}_{n\to\infty}\Big[4n\log 2+4\left\{\left(n+\frac{1}{2}\right)\log n-n+\mathrm{C}\right\}$$

$$-2\left\{\left(2n+\frac{1}{2}\right)\log(2n)-2n+\mathrm{C}\right\}-\log(2n+1)\Big]$$

$$=-2\log 2+2\mathrm{C},$$

so
$$\mathrm{C}=\tfrac{1}{2}\log(2\pi),$$

and finally

$$\log(n!) = \left(n+\frac{1}{2}\right)\log n - n + \frac{1}{2}\log 2\pi + \frac{1}{12n} - \frac{1}{360n^3} + \dots$$

or

$$n! = n^{n+\frac{1}{2}}\sqrt{2\pi}\cdot e^{-n}\left(1+\frac{1}{12n}+\dots\right).$$

This formula is due to Stirling:* it is of great value for computing the factorials of large numbers.

Ex. 1.—*To compute* $\log_{10}(79!)$.

In Stirling's formula, replacing $\log(n!)$ by $\{\log(n-1)!+\log n\}$ we obtain

$$\log_{10}(n-1)! = \left(n-\frac{1}{2}\right)\log_{10}n + \frac{1}{2}\log_{10}(2\pi)$$
$$+\mathrm{M}\left\{-n+\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\frac{1}{1680n^7}+\dots\right\}$$

where $\mathrm{M} = \log_{10}e = 0{\cdot}434294{,}481903{,}251828.$

Taking $n = 80$, $\log_{10}80 = 1{\cdot}903089{,}986991{,}943586$

and $79{\cdot}5\log_{10}80 = 151{\cdot}295653{,}965859{,}515$

then

$$\log_{10}(79!) = 79{\cdot}5\log_{10}80 + \frac{1}{2}\log_{10}(2\pi) + \mathrm{M}\left(\frac{1}{12.80}+\frac{1}{1260.80^5}\right)$$
$$-\mathrm{M}\left(80+\frac{1}{360.80^3}+\frac{1}{1680.80^7}\right)$$

$$\begin{array}{rr}
= 151{\cdot}295653{,}965859{,}515 & -\,34{\cdot}743558{,}552260{,}146 \\
0{\cdot}399089{,}934179{,}058 & -\,0{\cdot}000000{,}002356{,}198 \\
0{\cdot}000452{,}390085{,}316 & -\,0{\cdot}000000{,}000000{,}000 \\
0{\cdot}000000{,}000000{,}105 & \\
= 151{\cdot}695196{,}290123{,}994 & -\,34{\cdot}743558{,}554616{,}345 \\
= 116{\cdot}951637{,}735507{,}649. &
\end{array}$$

This is the required value correctly to fifteen places.

Ex. 2.—*Show that*

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = (f_{\frac{1}{2}}+f_{\frac{3}{2}}+f_{\frac{5}{2}}+\dots+f_{r-\frac{1}{2}}) + \frac{w}{24}(f_r'-f_0')$$
$$-\frac{7w^3}{5760}(f_r'''-f_0''') + \frac{31w^5}{967680}(f_r^{\mathrm{v}}-f_0^{\mathrm{v}}) - \dots$$

71. The Remainder Term in the Euler-Maclaurin Expansion.—We shall now investigate the error committed by truncating the Euler-Maclaurin expansion at any term.†

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* *Methodus differentialis* (1730), p. 137.

† Jacobi, *Journ. für Math.* 12 (1834), p. 263.

Let a set of polynomials $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, . . . be defined by the statement that $\phi_n(t)$ is the coefficient of $s^n/n!$ in the expansion of $s\cdot\frac{e^{ts}-1}{e^s-1}$ in ascending powers of s. We readily find

$$\phi_1(t)=t,\quad \phi_2(t)=t(t-1),\quad \phi_3(t)=t^3-\tfrac{3}{2}t^2+\tfrac{1}{2}t,\ \ldots$$

The polynomials $\phi_n(t)$ are called the *polynomials of Bernoulli.*[*]

From the defining equation

$$s\cdot\frac{e^{ts}-1}{e^s-1}=\sum_{n=1}^{\infty}\frac{\phi_n(t)s^n}{n!} \tag{1}$$

we see on putting $t=0$, and $t=1$ successively, *that all the Bernoullian polynomials vanish when $t=0$, and all of them after the first vanish when $t=1$.*

Moreover, by differentiating equation (1) with respect to t, we see that

$$\sum_{n=0}^{\infty}\frac{\phi'_{n+1}(t)s^n}{(n+1)!}-\sum_{n=1}^{\infty}\frac{\phi_n(t)s^n}{n!}=\frac{s}{e^s-1}.$$

Remembering that

$$\frac{s}{e^s-1}=1-\frac{s}{2}+\frac{B_1s^2}{2!}-\frac{B_2s^4}{4!}+\ldots,$$

and equating coefficients of powers of s, we have

$$\tfrac{1}{2}\phi_2'(t)=\phi_1(t)-\tfrac{1}{2}, \tag{2}$$
$$\tfrac{1}{3}\phi_3'(t)=\phi_2(t)+B_1, \tag{3}$$
$$\tfrac{1}{4}\phi_4'(t)=\phi_3(t), \tag{4}$$
$$\tfrac{1}{5}\phi_5'(t)=\phi_4(t)-B_2, \tag{5}$$

and so on.

Now denoting by $f(x)$ an arbitrary function, we have

$$\int_a^{a+w}f(x)dx=w\int_0^1 f(a+wt)dt$$
$$=\frac{w}{2}\int_0^1 f(a+wt)d(2t-1).$$

---

[*] The name was given by Raabe in honour of James Bernoulli, the author of *Ars Conjectandi.*

and, by repeated integration by parts,

$$\int_a^{a+w} f(x)dx = \frac{w}{2}\{f(a+w)+f(a)\} - \frac{w^2}{2}\int_0^1 (2t-1)f'(a+wt)dt$$

$$= \frac{w}{2}\{f(a+w)+f(a)\} - \frac{w^2}{2}\int_0^1 f'(a+wt)d\phi_2(t)$$

$$= \frac{w}{2}\{f(a+w)+f(a)\} + \frac{w^3}{2}\int_0^1 \phi_2(t)f''(a+wt)dt. \qquad (6)$$

But by (3) we have

$$\frac{w^3}{2}\int_0^1 \phi_2(t)f''(a+wt)dt$$

$$= -\frac{B_1 w^3}{2}\int_0^1 f''(a+wt)dt + \frac{w^3}{3!}\int_0^1 \phi_3'(t)f''(a+wt)dt$$

$$= -\frac{B_1 w^2}{2!}\{f'(a+w)-f'(a)\} - \frac{w^4}{3!}\int_0^1 f'''(a+wt)\phi_3(t)dt$$

$$= -\frac{B_1 w^2}{2!}\{f'(a+w)-f'(a)\} - \frac{w^4}{4!}\int_0^1 f'''(a+wt)\phi_4'(t)dt$$

$$= -\frac{B_1 w^2}{2!}\{f'(a+w)-f'(a)\} + \frac{w^5}{4!}\int_0^1 \phi_4(t)f^{\text{iv}}(a+wt)dt. \qquad (7)$$

On substituting from (7) in (6) we have

$$\int_a^{a+w} f(x)dx = \frac{w}{2}\{f(a+w)+f(a)\} - \frac{B_1 w^2}{2!}\{f'(a+w)-f'(a)\}$$

$$+ \frac{w^5}{4!}\int_0^1 \phi_4(t)f^{\text{iv}}(a+wt)dt. \qquad (8)$$

But by (5) we have

$$\frac{w^5}{4!}\int_0^1 \phi_4(t)f^{\text{iv}}(a+wt)dt$$

$$= \frac{B_2 w^5}{4!}\int_0^1 f^{\text{iv}}(a+wt)dt + \frac{w^5}{5!}\int_0^1 \phi_5'(t)f^{\text{iv}}(a+wt)dt$$

$$= \frac{B_2 w^4}{4!}\{f'''(a+w)-f'''(a)\} - \frac{w^6}{5!}\int_0^1 f^{\text{v}}(a+wt)\phi_5(t)dt$$

$$= \frac{B_2 w^4}{4!}\{f'''(a+w)-f'''(a)\} + \frac{w^7}{6!}\int_0^1 \phi_6(t)f^{\text{vi}}(a+wt)dt. \qquad (9)$$

Substituting from (9) in (8) we have

$$\int_a^{a+w} f(x)dx = \frac{w}{2}\{f(a+w)+f(a)\} - \frac{B_1 w^2}{2!}\{f'(a+w)-f'(a)\}$$

$$+ \frac{B_2 w^4}{4!}\{f'''(a+w)-f'''(a)\} + \frac{w^7}{6!}\int_0^1 \phi_6(t)f^{\text{vi}}(a+wt)dt.$$

This is the Euler-Maclaurin formula with a remainder-term expressing the error committed when the series is truncated at the terms involving the third derivatives. By proceeding in the same way we can find the error committed in truncating the Euler-Maclaurin expansion at any assigned term.

If $f(x)$ is a polynomial, the Euler-Maclaurin series terminates and is exact.

72. Gregory's Formula of Numerical Integration.— The earliest formula of numerical integration to be discovered was one in which the correction terms were expressed in terms of differences instead of derivatives. This formula, which is of great use in practice, may be derived from the Euler-Maclaurin formula in the following way:

In the Euler-Maclaurin formula

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \frac{1}{2}f_r\right) - \frac{w}{12}(f_r' - f_0') + \frac{w^3}{720}(f_r''' - f_0''') - \frac{w^5}{30240}(f_r^{\text{v}} - f_0^{\text{v}}) + \ldots$$

let us substitute for the derivatives from the equations

$$wf_0' = \Delta f_0 - \tfrac{1}{2}\Delta^2 f_0 + \tfrac{1}{3}\Delta^3 f_0 - \tfrac{1}{4}\Delta^4 f_0 + \tfrac{1}{5}\Delta^5 f_0 - \ldots$$

$$wf_r' = \Delta f_{r-1} + \tfrac{1}{2}\Delta^2 f_{r-2} + \tfrac{1}{3}\Delta^3 f_{r-3} + \tfrac{1}{4}\Delta^4 f_{r-4} + \tfrac{1}{5}\Delta^5 f_{r-5} + \ldots$$

$$w^3 f_0''' = \Delta^3 f_0 - \tfrac{3}{2}\Delta^4 f_0 + \tfrac{7}{4}\Delta^5 f_0 - \ldots$$

$$w^3 f_r''' = \Delta^3 f_{r-3} + \tfrac{3}{2}\Delta^4 f_{r-4} + \tfrac{7}{4}\Delta^5 f_{r-5} + \ldots$$

$$w^5 f_0^{\text{v}} = \Delta^5 f_0 - \ldots$$

$$w^5 f_r^{\text{v}} = \Delta^5 f_{r-5} + \ldots$$

etc.

It may be remarked that the formulae for the derivatives of f_0 are given in § 35, and that the formulae for the derivatives of f_r may be obtained by forming a difference-table in which the entries are reversed in order, *i.e.* $f_r, f_{r-1}, f_{r-2}, f_{r-3}, \ldots, f_1, f_0$, and applying the equations of § 35.

Thus we obtain

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \frac{1}{2}f_r\right) - \frac{1}{12}(\Delta f_{r-1} - \Delta f_0)$$
$$- \frac{1}{24}(\Delta^2 f_{r-2} + \Delta^2 f_0) - \frac{19}{720}(\Delta^3 f_{r-3} - \Delta^3 f_0) - \frac{3}{160}(\Delta^4 f_{r-4} + \Delta^4 f_0)$$
$$- \frac{863}{60480}(\Delta^5 f_{r-5} - \Delta^5 f_0) - \frac{275}{24192}(\Delta^6 f_{r-6} + \Delta^6 f_0) - \ldots$$

This formula was discovered by James Gregory:[*] like his formula of interpolation (§ 8), it does not require ordinates beyond the limits f_0 and f_r in order to form the differences.

On taking $f(x) = e^{vx}$, $w = 1$, $a = 0$, the formula becomes

$$\frac{e^{rv}-1}{v} = (1 + e^{v} + e^{2v} + \ldots + e^{(r-1)v}) + \frac{1}{2}(e^{rv} - 1) - \frac{1}{12}(1 - e^{-v})e^{rv}$$
$$+ \frac{1}{12}(e^{v} - 1) - \frac{1}{24}(1 - e^{-v})^2 e^{rv} - \frac{1}{24}(e^{v} - 1)^2 + \ldots,$$

which evidently breaks up into the expansions

$$\frac{1}{v} = \frac{1}{e^{v}-1} + \frac{1}{2} - \frac{1}{12}(e^{v} - 1) + \frac{1}{24}(e^{v} - 1)^2 - \frac{19}{720}(e^{v} - 1)^3 - \ldots$$

and

$$\frac{1}{v} = \frac{1}{1-e^{-v}} - \frac{1}{2} - \frac{1}{12}(1 - e^{-v}) - \frac{1}{24}(1 - e^{-v})^2 - \frac{19}{720}(1 - e^{-v})^3 - \ldots,$$

which are the expansions of

$$\frac{1}{\log\{1 + (e^{v} - 1)\}} \text{ and } -\frac{1}{\log\{1 - (1 - e^{-v})\}}$$

respectively. *The numerical coefficients in Gregory's formula are therefore the same as those which occur in the expansion of* $-\{\log(1 - x)\}^{-1}$ *in ascending powers of* x.

That this must be so may be seen at once by symbolic reasoning; for if D denotes the operation of differentiation, so that D^{-1} denotes integration, we have (§ 35)

$$e^{w\mathrm{D}} = \mathrm{E} = 1 + \Delta,$$

and therefore

$$\frac{1}{w}\mathrm{D}^{-1} = \frac{1}{\log(1 + \Delta)} = \frac{-1}{\log(1 - \Delta\mathrm{E}^{-1})},$$

whence Gregory's formula may be derived symbolically.

Expressions for the remainder after n terms of several formulae of numerical integration of this type have been found by H. P. Nielsen, *Arkiv för Math. Astron. och Fysik*, **4** (1908), Nr. 21.

Ex. 1.—*Show that the coefficient of* $-\{\Delta^n f_{r-n} \pm \Delta^n f_0\}$ *in Gregory's formula is*

$$(-1)^n \begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 & 0 & \ldots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 & \ldots \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad [(n + 1) \text{ rows}]$$

and that it may also be written

$$(-1)^n \int_0^1 \frac{x(x-1)(x-2)\ldots(x-n+1)}{n!}dx. \qquad \text{(Glaisher.)}$$

---

[*] Letter of Gregory to Collins of date 23rd November 1670, printed in Rigaud's *Correspondence*, **2**, p. 209.

Ex. 2.—*To calculate*

$$\int_{100}^{105}\frac{dx}{x}$$

correctly to seven places of decimals.

By Gregory's formula, the integral is equal to

$$\tfrac{1}{2}\tfrac{1}{100}+\tfrac{1}{101}+\tfrac{1}{102}+\tfrac{1}{103}+\tfrac{1}{104}+\tfrac{1}{2}\tfrac{1}{105}-\tfrac{1}{12}(\Delta f_{104}-\Delta f_{100})$$
$$-\tfrac{1}{24}(\Delta^2 f_{103}+\Delta^2 f_{100})-\tfrac{19}{720}(\Delta^3 f_{102}-\Delta^3 f_{100})-\dots \quad (1)$$

f.	Δ.	Δ².	Δ³.
$\frac{1}{100}=0{\cdot}010000\ 00$			
	-9901		
$\frac{1}{101}=0{\cdot}009900\ 99$		194	
	-9707		-5
$\frac{1}{102}=0{\cdot}009803\ 92$		189	
	-9518		-6
$\frac{1}{103}=0{\cdot}009708\ 74$		183	
	-9335		-6
$\frac{1}{104}=0{\cdot}009615\ 38$		177	
	-9158		
$\frac{1}{105}=0{\cdot}009523\ 81$			

Substituting in (1) the required differences, we have

$$\begin{array}{rlll}
\displaystyle\int_{100}^{105}\frac{dx}{x}= & 0{\cdot}005000\ 00 & -\tfrac{1}{12}(-9158+9901) & -\tfrac{1}{24}(177+194) \\
 & +0{\cdot}009900\ 99 & & \\
 & +0{\cdot}009803\ 92 & & \\
 & +0{\cdot}009708\ 74 & & \\
 & +0{\cdot}009615\ 38 & & \\
 & +0{\cdot}004761\ 90 & & \\
= & 0{\cdot}048790\ 93 & -62 & -15 \\
= & 0{\cdot}048790\ 16 & &
\end{array}$$

The required value is therefore 0·0487902 correctly to seven places.

Ex. 3.—*Given the table*

x.	$f(x)$.
1	287626,699801
2	287757,439208
3	287888,218227
4	288019,036864
5	288149,895125
6	288280,793016
7	288411,730543

compute $\int_1^7 f(x)dx$ *by the Gregory formula.*

Ex. 4.—*Calculate*

$$\int_{25}^{30}\frac{dx}{x}.$$

73. A Central-Difference Formula for Numerical Integration.—The formula of Gregory

$$\frac{1}{w}\int_a^{a+rw} f(x)dx=\left(\frac{1}{2}f_0+f_1+f_2+\ldots+f_{r-1}+\frac{1}{2}f_r\right)$$
$$-\frac{1}{12}(\Delta f_{r-1}-\Delta f_0)-\frac{1}{24}(\Delta^2 f_{r-2}+\Delta^2 f_0)-\frac{19}{720}(\Delta^3 f_{r-3}-\Delta^3 f_0)-\ldots$$

utilises differences which lie in lines sloping towards the centre of the difference table. Evidently there must exist a corresponding formula in which central-differences are used, and which will therefore be of the form

$$\begin{aligned}\frac{1}{w}\int_a^{a+rw} f(x)dx=&\left(\frac{1}{2}f_0+f_1+f_2+\ldots+f_{r-1}+\frac{1}{2}f_r\right)\\ &+\mathrm{A}\left\{\frac{\Delta f_r+\Delta f_{r-1}}{2}-\frac{\Delta f_0+\Delta f_{-1}}{2}\right\}\\ &+\mathrm{B}\{\Delta^2 f_{r-1}-\Delta^2 f_{-1}\}\\ &+\mathrm{C}\left\{\frac{\Delta^3 f_{r-1}+\Delta^3 f_{r-2}}{2}-\frac{\Delta^3 f_{-1}+\Delta^3 f_{-2}}{2}\right\}\\ &+\mathrm{D}\{\Delta^4 f_{r-2}-\Delta^4 f_{-2}\}\\ &+\ldots \qquad (1)\end{aligned}$$

where A, B, C, D are coefficients as yet unknown. To determine these coefficients, we may transform the derivatives in the Euler-Maclaurin formula, or more directly, we may write $f(x)=e^{vx}$, $w=1$ when formula (1) becomes

$$\frac{e^{(a+r)v}-e^{av}}{v}=\frac{e^{(a+r)v}-e^{av}}{e^v-1}+\frac{1}{2}\{e^{(a+r)v}-e^{av}\}+\mathrm{A}\sinh v\{e^{(a+r)v}-e^{av}\}$$
$$+\ldots$$

or

$$\frac{1}{v}=\frac{1}{e^v-1}+\frac{1}{2}+\mathrm{A}\sinh v+\ldots$$

or

$$\frac{1}{v}=\left(\frac{1}{v}-\frac{1}{2}+\frac{v}{12}-\frac{v^3}{720}+\ldots\right)+\frac{1}{2}+\mathrm{A}\left(v+\frac{v^3}{6}+\ldots\right)+\mathrm{B}(v^2+\ldots)$$
$$+\mathrm{C}(v^3+\ldots)+\ldots$$

Equating coefficients of powers of v we obtain

$$A = -\frac{1}{12}, \quad B = 0, \quad C = \frac{11}{720}, \ldots$$

and thus we have the formula

$$\begin{aligned}\frac{1}{w}\int_a^{a+rw} f(x)dx = &\left(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \frac{1}{2}f_r\right) \\ &-\frac{1}{12}\left\{\frac{\Delta f_r + \Delta f_{r-1}}{2} - \frac{\Delta f_0 + \Delta f_{-1}}{2}\right\} \\ &+\frac{11}{720}\left\{\frac{\Delta^3 f_{r-1} + \Delta^3 f_{r-2}}{2} - \frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2}\right\} + \ldots\end{aligned}$$

This formula, which is extensively used by astronomers in the calculation of perturbations, is in general more rapidly convergent than Gregory's; but it involves values of f outside the range of integration.

Ex. 1.—*Show that the above formula may be obtained by integrating with respect to n, between the limits* 0 *and* 1, *the interpolation-formula*

$$\begin{aligned}f(a+nw) = &\frac{f_0 + f_1}{2} + (n - \tfrac{1}{2})\Delta f_0 + \frac{n(n-1)}{2!}\frac{\Delta^2 f_{-1} + \Delta^2 f_0}{2} \\ &+ \frac{n(n-1)(n-\frac{1}{2})}{3!}\Delta^3 f_{-1} + \frac{n(n^2-1)(n-2)}{4!}\frac{\Delta^4 f_{-2} + \Delta^4 f_{-1}}{2} + \ldots\end{aligned}$$

Ex. 2.—*To calculate π from the formula*

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

Taking $w = \frac{1}{10}$, $r = 10$ we have

$$\begin{aligned}10\int_0^1 \frac{dx}{1+x^2} = &\frac{1}{2}\frac{1}{2\cdot00} + \frac{1}{1\cdot81} + \frac{1}{1\cdot64} + \frac{1}{1\cdot49} + \frac{1}{1\cdot36} + \frac{1}{1\cdot25} + \frac{1}{1\cdot16} + \frac{1}{1\cdot09} \\ &+ \frac{1}{1\cdot04} + \frac{1}{1\cdot01} + \frac{1}{2\cdot00} - \frac{1}{12}\left\{\frac{\Delta f_{10} + \Delta f_9}{2} - \frac{\Delta f_0 + \Delta f_{-1}}{2}\right\} \\ &+ \frac{11}{720}\left\{\frac{\Delta^3 f_9 - \Delta^3 f_8}{2} - \frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2}\right\} + \ldots\end{aligned}$$

[TABLE

f.	Δ.	Δ^2.	Δ^3.
$f_{-2}=\frac{1}{1{\cdot}04}=0{\cdot}9615385$			
	285605		
$f_{-1}=\frac{1}{1{\cdot}01}=0{\cdot}9900990$		-186595	
	99010		-11425
$f_0=\frac{1}{1{\cdot}00}=1{\cdot}0000000$		-198020	
	-99010		11425
$f_1=\frac{1}{1{\cdot}01}=0{\cdot}9900990$		-186595	
	-285605		
$f_2=\frac{1}{1{\cdot}04}=0{\cdot}9615385$			
. . . .			
. . . .			
$f_8=\frac{1}{1{\cdot}64}=0{\cdot}6097561$			
	-572699		
$f_9=\frac{1}{1{\cdot}81}=0{\cdot}5524862$		47837	
	-524862		1912
$f_{10}=\frac{1}{2{\cdot}00}=0{\cdot}5000000$		49749	
	-475113		-1162
$f_{11}=\frac{1}{2{\cdot}21}=0{\cdot}4524887$		48587	
	-426526		
$f_{12}=\frac{1}{2{\cdot}44}=0{\cdot}4098361$			

We have therefore, neglecting differences of order higher than the third,

$$10\int_0^1 \frac{dx}{1+x^2}=10\cdot\frac{\pi}{4}$$

$$\begin{array}{ll}
=0{\cdot}2500000 & -\frac{1}{12}(-499988-0)+\frac{11}{720}(375-0)\\
0{\cdot}5524862 & \\
0{\cdot}6097561 & \\
0{\cdot}6711409 & \\
0{\cdot}7352941 & \\
0{\cdot}8000000 & \\
0{\cdot}8620690 & \\
0{\cdot}9174312 & \\
0{\cdot}9615385 & \\
0{\cdot}9900990 & \\
0{\cdot}5000000 & \\
=7{\cdot}8498150 & +41666+6\\
=7{\cdot}8539822. &
\end{array}$$

Therefore

$$\pi = 4 \times 0{\cdot}78539822 = 3{\cdot}141592{,}88$$

whereas the correct value of π is $3{\cdot}141592{,}65$ . . .

74. Lubbock's Formula of Summation.—In the Euler-Maclaurin formula

$$\frac{1}{w}\int_a^{a+rw} f(x)dx = (f_0 + f_1 + f_2 + \ldots + f_r) - \frac{1}{2}(f_0 + f_r) - \frac{w}{12}(f_r' - f_0') + \frac{w^3}{720}(f_r''' - f_0''') - \ldots$$

suppose that each interval is subdivided into m parts: the Euler-Maclaurin formula corresponding to the division is

$$\frac{m}{w}\int_a^{a+rw} f(x)dx = (f_0 + f_{\frac{1}{m}} + f_{\frac{2}{m}} + \ldots + f_r) - \frac{1}{2}(f_0 + f_r) - \frac{w}{12m}(f_r' - f_0') + \frac{w^3}{720m^3}(f_r''' - f_0''') \ldots$$

Eliminating the integral between these two equations we have

$$f_0 + f_{\frac{1}{m}} + f_{\frac{2}{m}} + f_{\frac{3}{m}} + \ldots + f_r = m(f_0 + f_1 + f_2 + \ldots + f_r) - \frac{m-1}{2}(f_r + f_0) - \frac{(m^2-1)}{12m}w(f_r' - f_0') + \frac{m^4-1}{m^3}\frac{w^3}{720}(f_r''' - f_0''').$$

This formula enables us to deduce the sum of a large number of values of the function taken at any interval from a smaller number of values of the function, taken at intervals m times greater.

If the derivatives in this formula are replaced by their values in terms of differences, as in § 72, we have what is known as *Lubbock's formula*,*

$$\begin{aligned} f_0 + f_{\frac{1}{m}} + f_{\frac{2}{m}} + \ldots + f_r = m(f_0 + f_1 + \ldots + f_r) &- \frac{m-1}{2}(f_r + f_0) \\ &- \frac{m^2-1}{12m}(\Delta f_{r-1} - \Delta f_0) - \frac{(m^2-1)}{24m}(\Delta^2 f_{r-2} + \Delta^2 f_0) \\ &- \frac{(m^2-1)(19m^2-1)}{720m^3}(\Delta^3 f_{r-3} - \Delta^3 f_0) \\ &- \frac{(m^2-1)(9m^2-1)}{480m^3}(\Delta^4 f_{r-4} + \Delta^4 f_0) - \ldots \end{aligned}$$

---

* J. W. Lubbock, *Camb. Phil. Trans.* **3** (1829), p. 323. The formula as given originally by Lubbock involved advancing differences of f_r; the formula here given, which does not require a knowledge of f_x for values of x greater than r, is due to A. de Morgan, *Diff. Int. Calc.*, pp. 317-318, § 191. It may be obtained readily by the use of symbolic operators; cf. T. B. Sprague, *J.I.A.* **18** (1874), p. 309.

Ex. 1.—*To calculate the sum*

$$\frac{1}{100^2}+\frac{1}{101^2}+\frac{1}{102^2}+\ldots+\frac{1}{150^2}.$$

Let $m=10$, $w=10$, and form the difference table of $f(n)=\frac{1}{n^2}$, beginning with the value $f_0=\frac{1}{100^2}$.

$$\begin{array}{lrrrr}
f_0=0{\cdot}00010000 & & & & \\
 & -1736 & & & \\
f_1=0{\cdot}00008264 & & 416 & & \\
 & -1320 & & -123 & \\
f_2=0{\cdot}00006944 & & 293 & & 42 \\
 & -1027 & & -81 & \\
f_3=0{\cdot}00005917 & & 212 & & 26 \\
 & -815 & & -55 & \\
f_4=0{\cdot}00005102 & & 157 & & \\
 & -658 & & & \\
f_5=0{\cdot}00004444 & & & &
\end{array}$$

By Lubbock's formula we have

$$\begin{aligned}\sum_{100}^{150}\left(\frac{1}{n^2}\right) &= 10(f_0+f_1+\ldots+f_5)-\frac{9}{2}(f_5+f_0)-\frac{99}{120}(\Delta f_4-\Delta f_0)\\ &\quad -\frac{99}{240}(\Delta^2 f_3+\Delta^2 f_0)-\frac{99\times 1899}{720000}(\Delta^3 f_2-\Delta^3 f_0)\\ &\quad -\frac{99\times 899}{480000}(\Delta^4 f_1+\Delta^4 f_0)\\ &= 406710-64998-889{\cdot}35-236{\cdot}36-17{\cdot}76-12{\cdot}61\\ &= 340556.\end{aligned}$$

The required value is therefore 0·0034056.

Ex. 2.—*Evaluate to seven places*

$$\sum_{x=50}^{100}\frac{1}{x^2}.$$

Ex. 3.—*Show that Gregory's formula may be obtained by making* m *increase indefinitely in Lubbock's formula.*

75. Formulae which involve only selected Values of the Function.—If the function which is to be integrated is such that its differences beyond a certain order n may be neglected, the formula of Gregory (§ 72) enables us to express $\int_a^{a+rw} f(x)dx$ in terms of the values of f and its differences as far as order n, at selected values of x. But these differences can in turn be expressed in terms of selected values of f by reductions such as

$$\Delta f_0=f_1-f_0,\quad \Delta^2 f_0=f_2-2f_1+f_0, \text{ etc.,}$$

and therefore it is possible to express $\int_a^{a+rw} f(x)dx$ entirely in terms of values of f at selected values of x. Among formulae of this class may be mentioned specially the following,* which are accurate for functions whose fifth differences are constant:

$$\int_a^{a+6} f_x dx = \tfrac{3}{10}(f_a + 5f_{a+1} + f_{a+2} + 6f_{a+3} + f_{a+4} + 5f_{a+5} + f_{a+6}).$$

(Weddle.)

$$\int_a^{a+6} f_x dx = 0{\cdot}28(f_a + f_{a+6}) + 1{\cdot}62(f_{a+1} + f_{a+5}) + 2{\cdot}2f_{a+3}.$$

(G. F. Hardy.)

$$\int_a^{a+10} f_x dx = \tfrac{5}{126}\{8(f_a + f_{a+10}) + 35(f_{a+1} + f_{a+3} + f_{a+7} + f_{a+9})$$
$$+ 15(f_{a+2} + f_{a+4} + f_{a+6} + f_{a+8}) + 36f_{a+5})\}.$$

(Shovelton.)

These formulae may be proved directly, without reference to Gregory's formula, in the following way:

Let it be required, for example, to prove the second of them (G. F. Hardy's formula). Since f_x is a function whose fifth differences are constant, we can represent f_x by a polynomial of degree 5 in x, and therefore by a polynomial of degree 5 in the variable z, where $z = x - a - 3$, say

$$f_x = f_{a+3} + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5, \tag{1}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are independent of z.

Therefore

$$\int_a^{a+6} f_x dx = \int_{-3}^{3} (f_{a+3} + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \epsilon z^5)dz$$
$$= 6f_{a+3} + 18\beta + \frac{486}{5}\delta. \tag{2}$$

Now let us see if this can be represented by an expression of the type

$$A(f_a + f_{a+6}) + B(f_{a+1} + f_{a+5}) + Cf_{a+3}, \tag{3}$$

where A, B, C are absolute constants. By (1), (3) may be written

$$A(2f_{a+3} + 18\beta + 162\delta) + B(2f_{a+3} + 8\beta + 32\delta) + Cf_{a+3}. \tag{4}$$

Since the expressions (2) and (4) are to be identical, we may equate the coefficients of f_{a+3}, β, and δ in them, and thus obtain the equations

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* A valuable discussion of formulae of this type is given by W. F. Sheppard, *Proc. Lond. Math. Soc.* **32** (1900), p. 258 ; and a detailed elementary treatment by A. E. King, *Trans. Fac. Act.* **9** (1923), p. 218.

$$\begin{cases} 2A+2B+C=6 \\ 18A+8B=18 \\ 162A+32B=486/5. \end{cases}$$

These equations give

$$A=\frac{7}{25}=0{\cdot}28,\quad B=\frac{81}{50}=1{\cdot}62,\quad C=2{\cdot}2,$$

so (3) becomes

$$0{\cdot}28(f_a+f_{a+6})+1{\cdot}62(f_{a+1}+f_{a+5})+2{\cdot}2f_{a+3},$$

and thus Hardy's formula is established.

Weddle's formula may be deduced at once by adding the zero quantity $\frac{1}{50}\Delta^6 f_a$ to the right-hand side in Hardy's formula.

Ex. 1.—*Evaluate*

$$\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)}}.$$

Weddle's formula may be written

$$\int_a^{a+6w} f_x dx=\frac{3w}{10}(f_a+5f_{a+w}+f_{a+2w}+6f_{a+3w}+f_{a+4w}+5f_{a+5w}+f_{a+6w}).$$

Let $a=0$, $w=\frac{1}{12}$, and $f_x=1/\sqrt{(1-x^2)}$.

$$\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{(1-x^2)}}=\frac{1}{40}(1{\cdot}000000+5{\cdot}017452+1{\cdot}014185+6{\cdot}196773$$
$$1{\cdot}060660+5{\cdot}500191+1{\cdot}154700)$$
$$=0{\cdot}5235990.$$

The true value of this integral is $\frac{1}{6}\pi$ or 0·5235987 correctly to seven places.

Using Hardy's formula we have

$$\int_0^1\frac{dx}{\sqrt{(1-x^2)}}=\frac{1}{12}\{(0{\cdot}28\times 2{\cdot}1547005)+(1{\cdot}62\times 2{\cdot}1035285)$$
$$+2{\cdot}2\times 1{\cdot}0327955)\}=0{\cdot}5235985.$$

Ex. 2.—*Evaluate*

$$\int_0^1\frac{dx}{1+x^2},$$

using seven ordinates.

76. The Newton-Cotes Formulae of Integration.—Among formulae of the type discussed in the last section, the oldest and best known are the *Newton-Cotes formulae*, which we shall now derive from first principles.

Let it be required to calculate the integral of a given function between limits $x=a$ and $x=b$. By the transformation

$$x=\frac{a+b}{2}+\frac{b-a}{2}\xi$$

we can reduce the integral to one in which the limits are -1 and $+1$. Let $h_0, h_1, \ldots, h_n$ be numbers intermediate between -1 and $+1$: we shall try to find a sum of the type

$$H_0f(h_0)+H_1f(h_1)+\ldots+H_nf(h_n), \tag{1}$$

where $H_0, H_1, \ldots, H_n$ are constants, which will closely represent

$$\int_{-1}^{1} f(x)dx. \tag{2}$$

We are thus trying to represent the value of the integral by a kind of weighted average, in which weights $H_0, H_1, \ldots, H_n$ are attributed to the values of the ordinate $f(x)$, corresponding to the abscissae $h_0, h_1, \ldots, h_n$. In order to secure that the sum (1) shall represent closely the integral (2) we shall lay down the condition that the sum (1) is to be equal to the integral (2) *exactly* so long as $f(x)$ is any polynomial of degree less than $(n+1)$.

Denoting the product $(x-h_0)(x-h_1)\ldots(x-h_n)$ by $F(x)$, we see that the sum (1) exactly represents the integral (2) when

$$f(x)=\frac{F(x)}{x-h_r},$$

since this function is a polynomial of degree n. The sum (1) then reduces to

$$H_r(h_r-h_0)(h_r-h_1)\ldots(h_r-h_n),$$

where the factor (h_r-h_r) is omitted; and this may be written

$$H_r \operatorname*{Lt}_{x\to h_r}\frac{F(x)}{x-h_r}, \text{ or } H_rF'(h_r).$$

This gives at once

$$H_r=\frac{1}{F'(h_r)}\int_{-1}^{1}\frac{F(x)}{x-h_r}dx, \tag{3}$$

a formula which determines the H's when the h's are known.

This formula might have been derived by remarking that when $f(x)$ is a polynomial of degree less than $(n+1)$, it may be represented *accurately* by Lagrange's formula of interpolation (§ 17), which may be written

$$f(x) = \sum_r \frac{F(x)}{(x-h_r)F'(h_r)} f(h_r).$$

The required formula is obtained from this at once by integration.

Let us suppose that the h's are chosen at equal intervals apart so that

$$h_0 = -1,\ h_1 = -1 + \frac{2}{n},\ h_2 = -1 + \frac{4}{n},\ .\ .\ .,\ h_n = 1.$$

Then

$$\begin{aligned} \frac{F(x)}{x-h_r} &= (x-h_0)\ .\ .\ .\ (x-h_{r-1})\,(x-h_{r+1})\ .\ .\ .\ (x-h_n) \\ &= (x+1)\ldots\left\{x+1-\frac{2(r-1)}{n}\right\}\left\{x+1-\frac{2(r+1)}{n}\right\}\ldots(x-1) \\ &= \left(\frac{2}{n}\right)^n \{t(t-1)\ .\ .\ .\ (t-r+1)\,(t-r-1)\ .\ .\ .\ (t-n)\}, \end{aligned}$$

where $t = \frac{1}{2}n(x+1)$ and

$$\begin{aligned} F'(h_r) &= (h_r-h_0)\ .\ .\ .\ (h_r-h_{r-1})\,(h_r-h_{r+1})\ .\ .\ .\ (h_r-h_n) \\ &= (-1)^{n-r}\left(\frac{2}{n}\right)^n r\,!\,(n-r)\,! \end{aligned}$$

We have at once

$$\int_{-1}^{1} f(x)dx = H_0 f(-1) + H_1 f\left(-1+\frac{2}{n}\right) + \ .\ .\ .\ + H_n f(1),$$

where

$$H_r = \frac{(-1)^{n-r}2}{n.\,r\,!\,(n-r)\,!}\int_0^n t(t-1)\ldots(t-r+1)\,(t-r-1)\ldots(t-n)dt.$$

This is known as the Newton-Cotes formula of integration.[*]

More generally, writing $h_r = a + rw$, we have

$$\begin{aligned} \int_a^{a+nw} f(x)dx &= H_0 f(a) + H_1 f(a+w) \\ &\quad + H_2 f(a+2w) + \ .\ .\ .\ + H_n f(a+nw), \end{aligned}$$

---

[*] Newton, *Letter to Leibnitz* of date 24th October 1676; *Principia*, **3** (1687), Prop. xl. Lemma 5; Cotes, *Harmonia mensurarum* (1722); James Gregory (for $n=2$) in *Exercit. geom.* (1668). Cf. Bickley, *Math. Gazette*, **23** (1939) 352.

where

$$H_r = \frac{(-1)^{n-r}w}{r!\,(n-r)!}\int_0^n t(t-1)\ldots(t-r+1)(t-r-1)\ldots(t-n)dt.$$

The values of the coefficients H_r in a number of cases are given here for reference:

For $n=1$, $w=1$

$$H_0 = H_1 = \tfrac{1}{2}$$

For $n=2$, $w=\frac{1}{2}$

$$H_0 = H_2 = \tfrac{1}{6}, \quad H_1 = \tfrac{2}{3}$$

For $n=3$, $w=\frac{1}{3}$

$$H_0 = H_3 = \tfrac{1}{8}, \quad H_1 = H_2 = \tfrac{3}{8}$$

For $n=4$, $w=\frac{1}{4}$

$$H_0 = H_4 = \tfrac{7}{90}, \quad H_1 = H_3 = \tfrac{16}{45}, \quad H_2 = \tfrac{2}{15}$$

For $n=5$, $w=\frac{1}{5}$

$$H_0 = H_5 = \tfrac{19}{288}, \quad H_1 = H_4 = \tfrac{25}{96}, \quad H_2 = H_3 = \tfrac{25}{144}$$

For $n=6$, $w=\frac{1}{6}$

$$H_0 = H_6 = \tfrac{41}{840}, \quad H_1 = H_5 = \tfrac{9}{35}, \quad H_2 = H_4 = \tfrac{9}{280}, \quad H_3 = \tfrac{34}{105}$$

Note that we must always have $H_0 + H_1 + H_2 + \ldots + H_n = nw$.

A formula making use of the ordinates at the middle of the intervals (instead of the ends of the intervals) was given by Maclaurin, *Fluxions*, 2 (1742), p. 832.

An expression for the remainder after n terms in the Newton-Cotes formula has been given by J. F. Steffensen, *Skandinavisk Aktuarietidskrift* (1921), p. 201.

Ex.—Calculate

$$\int_0^1 \frac{dx}{1+x}$$

by the Newton-Cotes formula with eight ordinates.

Let $w=\frac{1}{8}$ so that $n=8$, and write

$$\phi(t) = t(t-1)\ldots(t-8).$$

The coefficients H_r may be written

$$H_r = \frac{(-1)^{8-r}}{r!\,(8-r)!}\,\frac{1}{8}\int_0^8 \frac{\phi(t)dt}{t-r},$$

and we have the values

$$H_0 = H_8 = \tfrac{989}{28350} : H_1 = H_7 = \tfrac{2944}{14175} : H_2 = H_6 = -\tfrac{464}{14175} :$$
$$H_3 = H_5 = \tfrac{5248}{14175} : H_4 = -\tfrac{454}{2835}.$$

Since $f(x) = \dfrac{1}{1+x}$, we have

x	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
$f(x)$	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$
$14175H$	494·5	2944	-464	5248	-2270	5248	-464	2944	494·5

Then

$$\int_0^1 \frac{dx}{1+x} = \mathrm{H}_0 \cdot 1 + \mathrm{H}_1 \cdot \frac{8}{9} + \mathrm{H}_2 \cdot \frac{4}{5} + \mathrm{H}_3 \cdot \frac{8}{11} + \mathrm{H}_4 \cdot \frac{2}{3} + \mathrm{H}_5 \cdot \frac{8}{13} + \mathrm{H}_6 \cdot \frac{4}{7} + \mathrm{H}_7 \cdot \frac{8}{15} + \mathrm{H}_8 \cdot \frac{1}{2}$$

$$= \frac{1}{14175}\left\{494 \cdot 5 \times 1 \cdot 5 + \frac{192}{135} \cdot 2944 - \frac{48}{35} \cdot 464 + \frac{192}{143} \cdot 5248 - \frac{2}{3} \cdot 2270\right\}$$

$= 0 \cdot 69314721$,

whereas the true value is 0·69314718.

77. The Trapezoidal and Parabolic Rules.—The simplest cases of the Newton-Cotes formula are the following:

1°. $n = 1$.

$$\int_a^{a+w} f(x)dx = \tfrac{1}{2}wf(a) + \tfrac{1}{2}wf(a+w).$$

This is known as the *trapezoidal rule.* It is exact when $f(x)$ is a function whose first differences are constant. When $f(x)$ is a function whose first differences are not constant, the difference between the two sides of the above equation may be written

$$\tfrac{1}{12}w^3 f''(a + \theta w),$$

where $0 < \theta < 1$.

2°. $n = 2$.

$$\int_a^{a+2w} f(x)dx = \tfrac{1}{3}wf(a) + \tfrac{4}{3}wf(a+w) + \tfrac{1}{3}wf(a+2w).$$

This formula was first given (in a geometrical form) by Cavalieri,* and later by James Gregory † and by Thomas Simpson ‡; it is generally known as *Simpson's* or the *parabolic* rule. It is exact when $f(x)$ is a function whose third differences are constant; when $f(x)$ is a function whose third differences are not constant, the error involved in the use of Simpson's formula, *i.e.* the difference between the first and second members, may be expressed in the form §

$$-\frac{w^5}{90} f^{\mathrm{iv}}(a + 2\theta w),$$

where $0 < \theta < 1$.

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* *Centuria di varii problemi,* Bologna (1639), p. 446.

† *Exercit. geom.*, London (1668).

‡ *Math. Dissertations,* London (1743), p. 109.

§ Peano, *Applicazioni geom. del calcolo infin.*, Turin (1887), p. 208.

3°. $n=3$.

$$\int_a^{a+3w} f(x)dx = \tfrac{3}{8}w\{f(a)+3f(a+w)+3f(a+2w)+f(a+3w)\}.$$

This is generally called the *three-eighths rule.*

In practice, when it is required to compute an integral, we first divide the range of integration into a number of intervals and then apply one of the above rules to each of the intervals and add the results so obtained. Thus when we use Simpson's rule after dividing the range (say from a to $a+2pw$) into p intervals, we have for the value of the integral

$$\begin{aligned}&\tfrac{1}{3}wf(a)+\tfrac{4}{3}wf(a+w)+\tfrac{1}{3}wf(a+2w)\\ &+\tfrac{1}{3}wf(a+2w)+\tfrac{4}{3}wf(a+3w)+\tfrac{1}{3}wf(a+4w)\\ &+\tfrac{1}{3}wf(a+4w)+\tfrac{4}{3}wf(a+5w)+\tfrac{1}{3}wf(a+6w)\\ &+\ldots\end{aligned}$$

or

$$\begin{aligned}&\tfrac{1}{3}w[f(a)+2\{f(a+2w)+f(a+4w)+\ldots\}\\ &\qquad\qquad+4\{f(a+w)+f(a+3w)+\ldots\}+f(a+2pw)],\end{aligned}$$

whence the form in which Simpson's rule is generally stated: *Form the sum of the extreme ordinates, twice the sum of the even ordinates, and four times the sum of the odd ordinates; and multiply the result by one-third of the interval of the abscissa.*

Ex. 1.—*Calculate*

$$\int_0^1 \frac{dx}{1+x}$$

by the Simpson formula, using nine ordinates.

Let $f(x)=\dfrac{1}{1+x}$, $a=0$, and $w=\dfrac{1}{8}$. We have the values

x	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
$f(x)$	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$

Then

$$\begin{aligned}\int_0^1 \frac{dx}{1+x} &= \frac{1}{24}\left\{\left(1+\frac{1}{2}\right)+2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)+4\left(\frac{8}{9}+\frac{8}{11}+\frac{8}{13}+\frac{8}{15}\right)\right\}\\ &=\frac{1}{24}\{1{\cdot}5+4{\cdot}076190\,5+11{\cdot}059518\,3\}\\ &=0{\cdot}693154\,5.\end{aligned}$$

The value obtained is 0·693155, whereas the correct value is $\log 2 = 0{\cdot}693147$, correctly to six places.

Ex. 2.—*Show that if in evaluating the area between the ordinates* $x=1$ *and* $x=2$ *of the rectangular hyperbola* $xy=1$ *the formula of Simpson with* 11 *ordinates is employed, the error is* 0·000003.

78.—Woolhouse's Formulae.—A number of integration formulae akin to those which have been considered in the last three articles were given by Woolhouse in *Journ. Inst. Act.* 27 (1888), p. 122. He recommended the two following specially for their "high and persistent approximation."

$$\int_0^{10} udt = 5\left\{\frac{223}{3969}(u_0+u_{10})+\frac{5875}{18144}(u_1+u_9)+\frac{4625}{10584}(u_3+u_7)+\frac{41}{112}u_5\right\}$$

$$= [9\cdot4485938](u_0+u_{10})+[0\cdot2092449](u_1+u_9) + [0\cdot3394319](u_3+u_7)+[0\cdot2625358]u_5$$

(where the numbers in square brackets are the logs of the coefficients), and

$$\int_0^{28} udt = 14\left\{\frac{7}{195}(u_0+u_{28})+\frac{16807}{66690}(u_2+u_{26})+\frac{128}{285}(u_7+u_{21})+\frac{71}{135}u_{14}\right\}$$

$$= [9\cdot7011914](u_0+u_{28})+[0\cdot5475575](u_2+u_{26}) + [0\cdot7984931](u_7+u_{21})+[0\cdot8670526]u_{14}.$$

79.—Chebyshef's Formulae.—If, instead of laying down the condition in formula (1) of § 76 that the intervals h_1-h_0, h_2-h_1, etc., are to be equal, we lay down the condition that the coefficients H_0, H_1, . . ., H_n are to be equal to each other, we obtain a set of formulae first given by Chebyshef,* which have found acceptance chiefly with naval architects. They have a certain advantage when the ordinates are experimental data liable to unknown errors; for if we have a number of quantities which are equally liable to be affected with error, and if a linear function of these quantities is formed, the sum of the coefficients in this function being fixed, then the probable error of the function is least when the coefficients are all equal.

Chebyshef's general formula is that if $f(x)$ is a function whose $(n+1)$th differences are negligible, then

$$\int_{-1}^{1} f(x)dx = \frac{2}{n}\{f(x_1)+f(x_2)+\ .\ .\ .\ +f(x_n)\},$$

where x_1, x_2, . . ., x_n are the roots of a certain polynomial, which is, in fact, the polynomial part of the expansion of

$$x^n e^{-\frac{n}{2.3x^2}-\frac{n}{4.5x^4}-\frac{n}{6.7.x^6}-\ .\ .\ .}$$

in descending powers of x.

Thus, when $f(x)$ is a function whose sixth differences are negligible, we have

$$\int_{-1}^{1} f(x)dx = \tfrac{2}{5}\{f(x_1)+f(x_2)+f(x_3)+f(x)_4+f(x_5)\},$$

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* *Journ. de math.* (2), 19 (1874), p. 19: *Assoc. Franç.* 2 (Lyon, 1873), p. 69.

where $x_1, x_2, \ldots, x_5$ are the roots of the equation

$$x^5 - \tfrac{5}{6}x^3 + \tfrac{7}{72}x = 0,$$

so that

$$\begin{aligned} -x_1 &= 0{\cdot}832437 = x_5 \\ -x_2 &= 0{\cdot}374542 = x_4, \\ x_3 &= 0. \end{aligned}$$

Chebyshef's formulae were suggested by a formula due to Bronwin,* namely

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{(1-x^2)}} = \int_0^\pi f(\cos t)dt$$

$$= \frac{\pi}{n}\left\{f\left(\cos\frac{\pi}{2n}\right) + f\left(\cos\frac{3\pi}{2n}\right) + f\left(\cos\frac{5\pi}{2n}\right) + \ldots + f\left(\cos\frac{(2n-1)\pi}{2n}\right)\right\},$$

which is rigorously true when $f(x)$ is a function whose $2n$th differences are zero. Thus if $f(x)$ is a function whose sixth differences are negligible, we have

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{(1-x^2)}} = \frac{\pi}{3}\left\{f\left(\frac{\sqrt{3}}{2}\right) + f(0) + f\left(-\frac{\sqrt{3}}{2}\right)\right\}.$$

80. **Gauss's Formula of Numerical Integration.**—Suppose now that instead of prescribing the numbers $h_0, h_1, \ldots, h_n$ of § 76 by the condition that they are to be at equal intervals apart, we determine them by the condition that the formula is to have the highest possible degree of accuracy; since there are at our disposal $(2n+2)$ constants $h_0, h_1, \ldots, h_n, \mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_n$, we can choose them so as to make the formula rigorously true when $f(x)$ is any polynomial of degree $(2n+1)$. Denoting the product $(x-h_0)(x-h_1) \ldots (x-h_n)$ as before by $\mathrm{F}(x)$, we must therefore have in particular

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \phi(x)\mathrm{F}(x)dx = 0,$$

where $\phi(x)$ denotes any polynomial of degree less than $(n+1)$. Now it is a known property† of the Legendre polynomial $\mathrm{P}_{n+1}(x)$ that

$$\int_{-1}^{1} \phi(x)\mathrm{P}_{n+1}(x)dx = 0,$$

so long as $\phi(x)$ is any polynomial of degree less than $(n+1)$. We see, therefore, that the stated conditions are satisfied by taking

$$\mathrm{F}(x) = \mathrm{P}_{n+1}(x),$$

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* *Phil. Mag.* **34** (1849), p. 262.

† Whittaker and Watson, *Modern Analysis*, § 15·14.

so the numbers $h_0, h_1, \ldots, h_n$ are the roots of the Legendre polynomial of degree $(n+1)$.* The H's are then given by formula (3) of § 76.

In the simplest cases we obtain the following values for the h's and H's:

$$n=1,\quad h_0 = -\frac{1}{\sqrt{3}},\quad h_1 = \frac{1}{\sqrt{3}},\quad \mathrm{H}_0 = 1,\quad \mathrm{H}_1 = 1.$$

$$n=2,\quad h_0 = -\frac{\sqrt{3}}{\sqrt{5}},\quad h_1 = 0,\quad h_2 = \frac{\sqrt{3}}{\sqrt{5}},\quad \mathrm{H}_0 = \frac{5}{9},\quad \mathrm{H}_1 = \frac{8}{9},\quad \mathrm{H}_2 = \frac{5}{9}.$$

$$n=3,\quad h_0^2 = h_3^2 = \frac{3}{7} + \frac{2}{7}\frac{\sqrt{6}}{\sqrt{5}},\quad h_1^2 = h_2^2 = \frac{3}{7} - \frac{2}{7}\frac{\sqrt{6}}{\sqrt{5}},$$

$$\mathrm{H}_0 = \mathrm{H}_3 = \frac{18-\sqrt{30}}{36},\quad \mathrm{H}_1 = \mathrm{H}_2 = \frac{18+\sqrt{30}}{36}.$$

Converting to decimals and transforming the variable so that the range of integration is from 0 to 1, we have

$$\int_0^1 \psi(x)dx = \mathrm{A}_0\psi(x_0) + \mathrm{A}_1\psi(x_1) + \ldots + \mathrm{A}_n\psi(x_n),$$

where x_r and A_r are the transformed values of h_r and H_r respectively, and in fact

$n=1$

$x_0 = 0{\cdot}211324\ 87$ $\qquad \mathrm{A}_0 = \mathrm{A}_1 = 0{\cdot}5$
$x_1 = 0{\cdot}788675\ 13$

$n=2$

$x_0 = 0{\cdot}112701\ 67$ $\qquad \mathrm{A}_0 = \mathrm{A}_2 = \frac{5}{18}$
$x_1 = 0{\cdot}5$ $\qquad \mathrm{A}_1 = \frac{4}{9}$
$x_2 = 0{\cdot}887298\ 33$

$n=3$

$x_0 = 0{\cdot}069431\ 84$ $\qquad \mathrm{A}_0 = \mathrm{A}_3 = 0{\cdot}173927\ 4$
$x_1 = 0{\cdot}330009\ 48$ $\qquad \mathrm{A}_1 = \mathrm{A}_2 = 0{\cdot}326072\ 6$
$x_2 = 0{\cdot}669990\ 52$
$x_3 = 0{\cdot}930568\ 16$

$n=4$

$x_0 = 0{\cdot}046910\ 08$ $\qquad \mathrm{A}_0 = \mathrm{A}_4 = 0{\cdot}118463\ 4$
$x_1 = 0{\cdot}230765\ 34$ $\qquad \mathrm{A}_1 = \mathrm{A}_3 = 0{\cdot}239314\ 3$
$x_2 = 0{\cdot}5$ $\qquad \mathrm{A}_2 = 0{\cdot}284444\ 4$
$x_3 = 0{\cdot}769234\ 66$
$x_4 = 0{\cdot}953089\ 92$

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* Gauss, "Methodus nova integralium valores per approx. inveniendi," *Gött. Comm.* III. (1814) = *Werke*, 3, p. 163.

Ex. 1.—*Find by Gauss's method, with $n=4$, the value of*

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx.$$

Writing

$$\psi(x)=\frac{\log(1+x)}{1+x^2}=2{\cdot}302585\,09\frac{\log_{10}(1+x)}{1+x^2},$$

we may arrange the working in the form

	x_0.	x_1.
$1+x$	1·046910 08	1·230765 34
$\log_{10}(1+x)$	0·019909 38	0·090175 26
A	0·118463 4	0·284444 4
$1+x^2$	1·002200 56	1·053252 64
$\log_{10} 2{\cdot}302585\,09$	0·362215 69	0·362215 69
$\log_{10}\{\log_{10}(1+x)\}$	8·299057 73	8·955087 40
$\log_{10}$ A	9·073584 20	9·378968 65
	17·734857 62	18·696271 74
$\log_{10}(1+x^2)$	0·000954 64	0·022532 56
$\log_{10} A\psi(x)$	7·733902 98	8·673739 18
$A\psi(x)$	0·005418 80	0·047177 96

We thus obtain the values

$$\begin{aligned}A_0\psi(x_0)&=0{\cdot}005418\,80\\ A_1\psi(x_1)&=0{\cdot}047177\,96\\ A_2\psi(x_2)&=0{\cdot}092265\,82\\ A_3\psi(x_3)&=0{\cdot}085781\,36\\ A_4\psi(x_4)&=0{\cdot}041554\,03.\end{aligned}$$

Finally we have

$$\begin{aligned}\int_0^1\frac{\log(1+x)}{1+x^2}dx&=A_0\psi(x_0)+A_1\psi(x_1)+\ldots+A_4\psi(x_n)\\ &=0{\cdot}272198\,0,\end{aligned}$$

whereas the correct value of this integral $\left(=\frac{\pi}{8}\log 2\right)$ is 0·272198 3.

Ex. 2.—*Show that a degree of accuracy equal to that of Gauss's formula is obtained by the use of*

$$\int_{-1}^1 F(x)dx=\frac{1}{n(n+1)}\Sigma\frac{F(h)}{\{P_n(h)\}^2},$$

the sum being extended over all the roots h of the equation

$$P_{n+1}(x)-P_{n-1}(x)=0,$$

that is, over + 1, − 1, *and the roots of* $P_n'(x)=0$.

(Radau.)

MISCELLANEOUS EXAMPLES ON CHAPTER VII

1. Show that if fifth differences are negligible

$$\int_a^{a+6n} f_x dx = 0{\cdot}28(f_a + 2f_{a+6} + 2f_{a+12} + \cdot\cdot\cdot + f_{a+6n})$$
$$+ 1{\cdot}62(f_{a+1} + f_{a+3} + \cdot\cdot\cdot) + 0{\cdot}58(f_{a+3} + f_{a+9} + \cdot\cdot\cdot).$$

(Shovelton.)

2. Show that if fifth differences are negligible

$$\int_a^{a+6n} f_x dx = \tfrac{3}{10}[(f_a + f_{a+2} + f_{a+4} + \cdot\cdot\cdot + f_{a+6n})$$
$$+ 5(f_{a+1} + f_{a+3} + \cdot\cdot\cdot + f_{a+6n-1}) + (f_{a+3} + f_{a+6} + \cdot\cdot\cdot + f_{a+6n-3})].$$

(Shovelton.)

3. Show that the integral

$$\int_{-1}^{1} \phi(x)\,(1-x)^\lambda (1+x)^\mu dx$$

may be represented by the sum

$$H_0\phi(h_0) + H_1\phi(h_1) + \cdot\cdot\cdot$$

exactly so long as $\phi(x)$ is a polynomial of degree less than $2n$, provided the abscissae h_0, h_1, . . . are the roots of the equation $F(x) = 0$, where $F(x)$ is defined by the relation

$$(x-1)^\lambda (x+1)^\mu F(x) = \frac{d^n}{dx^n}\{(x-1)^{n+\lambda}(x+1)^{n+\mu}\},$$

and the H's are defined by

$$H_r = \frac{1}{F'(h_r)}\int_{-1}^{1} \frac{F(x)f(x)}{x-h_r}dx.$$

(Radau.)

4. The values of a function $f(x)$ are given by the table

x.	$f(x)$.
1·050	1·25386
1·060	1·26996
1·070	1·28619
1·080	1·30254
1·090	1·31903
1·100	1·33565.

Using the Gregory formula, calculate

$$\int_{1\cdot050}^{1\cdot100} f(x)dx.$$

5. The values of a function $f(x)$ are given by the table

x.	$f(x)$.
41	395 254 161
42	406 586 896
43	418 161 601
44	429 981 696
45	442 050 625
46	454 371 856
47	466 948 881.

Evaluate
$$\int_{41}^{47} f(x)dx.$$

6. If $f(x)$ is a function whose fourth differences are negligible, prove that

$$\int_0^\infty e^{-x}f(x)dx = \frac{2+\sqrt{2}}{4}f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4}f(2+\sqrt{2}),$$

and, more generally, if $f(x)$ is a function whose $2n$th differences are negligible, obtain a formula

$$\int_0^\infty e^{-x}f(x)dx = \sum_{k=1}^{n} A_k f(x_k),$$

where $x_1, x_2, \ldots, x_n$ are the roots of the polynomial $e^x\frac{d^n}{dx^n}(e^{-x}x^n)$.

(A. Berger.)

7. If $f(x)$ is a function whose sixth differences are negligible, prove that

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)dx = \frac{\sqrt{\pi}}{6}\left\{f\left(-\sqrt{\frac{3}{2}}\right) + 4f(0) + f\left(\sqrt{\frac{3}{2}}\right)\right\},$$

and more generally, if $f(x)$ is a function whose $2n$th differences are negligible, obtain a formula

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)dx = \sum_{k=1}^{n} A_k f(x_k),$$

where $x_1, x_2, \ldots, x_n$ are the roots of the polynomial $e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$.

(A. Berger.)

8. Show that Hardy's formula (p. 151) may be derived from Gauss's general formula by taking the nearest integral values for the ordinates.

ADDITIONAL REFERENCES

V. Cassina, "Formole sommatorie e di quadratura ad ordinate estreme", *Rend. Ist. Lombardo,* **72** (1939), p. 1.

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CHAPTER VIII

NORMAL FREQUENCY DISTRIBUTIONS

81. Frequency Distributions.—In statistical investigations attention is directed to a group of persons or objects, and an enumeration is made of the individuals in the group who have some particular attribute: for instance, we may consider the group consisting of all the inhabitants of Edinburgh and enumerate those who are widowers. In the most important and interesting cases the attribute is one which is capable of being expressed by a number; thus the attribute might be the height of the individual, which is expressible as a number of inches. In such cases the whole group may be partitioned into classes or sub-groups according to the value of this number: thus the whole group of the inhabitants of Edinburgh may be arranged in classes according to the number of inches in their stature; one class, for example, might consist of persons whose height is between $68\frac{1}{2}$ inches and $69\frac{1}{2}$ inches. Now let x_1 be the number of inches in the height of the shortest person in the city, and let the number of persons whose height is between $x_1 - \frac{1}{2}$ and $x_1 + \frac{1}{2}$ inches be y_1; let $x_1 + 1$ be denoted by x_2, and let the number of persons whose height is between $x_2 - \frac{1}{2}$ and $x_2 + \frac{1}{2}$ be y_2; let $x_2 + 1$ be denoted by x_3, and let the number of persons whose height is between $x_3 - \frac{1}{2}$ and $x_3 + \frac{1}{2}$ be denoted by y_3, and so on. Then we may regard $x_1, x_2, x_3, \ldots$ as successive values of an argument and $y_1, y_2, y_3, \ldots$ as the corresponding values of a function of this argument. The table of values thus obtained specifies the distribution of the heights among the inhabitants of Edinburgh: such a distribution is called a *frequency distribution.*

As an example of a frequency distribution, we may take the following results of measurements of the chests of 5732 soldiers in Scottish regiments.*

Chest measure in inches	33	34	35	36	37	38	39	40
Number of men . .	3	19	81	189	409	753	1062	1082

Chest measure in inches	41	42	43	44	45	46	47	48
Number of men . .	935	646	313	168	50	18	3	1

The type of frequency distribution which is most familiar to the worker in experimental science is the distribution of the measures obtained by repeated measurements of the same observed quantity. Let the true measure of an observed quantity be a. Since measurements made of this quantity by the same or different observers are affected by errors of observation, the measures actually obtained will not all be equal to a, but will form a group of measures $a_1, a_2, a_3, a_4, \ldots$ not differing greatly from a. The practical problem is to obtain the best possible estimate for a when we know the measures $a_1, a_2, a_3, \ldots$, and also to estimate the error to which this value is liable. In order to solve these problems, we must study the type of frequency distribution to which the group $a_1, a_2, a_3, \ldots$ belongs.

82. **Continuous Frequency Distributions.**—In the above numerical example we have grouped together in a single class all the 753 men whose chest measure is between 37½ and 38½ inches. Supposing that full information regarding the individual measures is at our disposal, we might have divided this class into two classes, one consisting of men whose chest measure is between 37½ and 38 inches, and the other whose chest measure is between 38 and 38½ inches; and in this way we might subdivide each of the original classes, thereby evidently doubling the total number of classes and so producing a more detailed statement regarding the chest measures of the men. Further, we might divide each of the new classes into two, thus quadrupling the original number of classes. But evidently if we attempt to proceed very far in this direction we shall meet with practical difficulties: thus no statistician would think of trying to arrange the men in classes each of

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* *Edinburgh Medical Journal,* **13** (1817), p. 260.

which comprehends a variation of only one-thousandth of an inch in height, partly because the measures cannot be relied on to such great accuracy, and partly because the numbers of men in the classes would become small and irregular, and would cease to present to the mind a clear picture of the frequency distribution.

For theoretical purposes, however, we may disregard these practical difficulties and consider an ideal case in which the measures are supposed perfectly accurate, and the number of individuals in the whole group is supposed very large, so that, however narrow we make the qualification for membership of a class, there will always be enough members in it to make the sequence of the numbers of members of classes a regular sequence. Now consider the class constituted of individuals whose measure is between x and $x+\epsilon$, where ϵ is a small number. The number of members in this class will evidently be approximately proportional to ϵ, since doubling the range of qualification for the class would approximately double the membership; and it will also be proportional to the number N of individuals in the whole group, since, if twice as many soldiers were measured, the number of soldiers in each class would be approximately doubled. Let us then denote the number of individuals whose measure is between x and $x+\epsilon$ by $\mathrm{N}\epsilon y$; then the number y depends on x, and in fact the way in which y depends on x specifies completely the frequency distribution. We shall often express the dependence of y on x by writing $\phi(x)$ for y, and we shall use the differential notation, writing dx for ϵ, so that in a group of N individuals the number whose measure is between x and $x+dx$ is denoted by $\mathrm{N}\phi(x)dx$.

Expressing the same idea in other words, we may say that $\phi(x)dx$ represents the *probability* that an individual chosen at random in the whole group has a measure between x and $x+dx$; and hence if the *frequency curve* $y=\phi(x)$ be drawn, the measure of any individual is *equally likely* to be the abscissa of any point taken at random within the area bounded by the curve $y=\phi(x)$ and the axis of x.

The earliest mathematical discussion of a frequency distribution seems to have been that of Simpson (1756) in connection with the Theory of

Errors of Observation. Simpson assumed that the probability of a positive error lying between x and $x+dx$ was $\phi(x)dx$, where $\phi(x) = -mx+c$, m and c being positive constants: the probability of a negative error was assumed equal to the probability of a positive error of the same amount, and errors of greater magnitude than c/m were assumed to be impossible. The frequency curve for the errors is therefore, in this case, two sides of an isosceles triangle, together with the prolongations of the base outside the triangle.

83. **Basis of the Theory of Frequency Distributions.**—We shall now approach the discussion of frequency curves from the theoretical side. The considerations on which the theoretical investigation is based were first put forward by Laplace, and may be described as follows.

Consider the frequency distribution of (say) the chest measures of the soldiers (§ 81). If we fix our attention on an individual soldier, we may say that a great many different factors have contributed to make his chest measure what it actually is. For, owing to heredity, it will have been influenced by the chest measures of his father and mother; but the chest measure of his father will in turn have been influenced by those of *his* father and mother, and so on, so that ultimately the chest measure of the individual soldier we are considering may be regarded as influenced by the chest measures of a very large number of individuals of a remote generation, each of them singly making only a very small contribution to the total effect. Moreover, other factors will enter: for example, his nourishment and exercise at different ages of life. Thus *the deviation of the actual chest measure of an individual from the average may be regarded as the sum of a very great number of very small deviations (positive or negative) due to the separate factors in the heredity and environment of the individual.* These multitudinous small component deviations will be assumed to be *independent of* each other, in the sense in which the word "independent" is used in the Theory of Probability.

84. **Galton's Quincunx.**—An interesting piece of apparatus was devised by Galton* to illustrate the formation of a frequency distribution from the joint effect of a large number of small and independent deviations.

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* *Natural Inheritance*, p. 63; for experimental results cf. O. Gruber, *Zeits. für Math. u. Phys.* **56** (1908), p. 322.

Into a board inclined to the horizontal about a thousand pins are driven, disposed in the fashion known to fruit growers as the *quincunx, i.e.* so that every pin forms equilateral triangles with its nearest neighbours. At the top of the board is a funnel, into which small shot is poured. The shot in descending strikes the pins in the successive rows, each piece being deviated to right or to left at every encounter with a pin. At the bottom of the board are about thirty compartments into which the shot ultimately falls. It is found that the middle compartment receives most shot, and that the falling-off in the amount of shot received by the compartments, as we proceed outwards from the middle compartment, resembles closely the falling off in the number of men having corresponding deviations from the average chest measures (§ 81), or the falling off in the number of measurements which have corresponding deviations from the average of the measures of an observed quantity. In fact, the curve formed by the outline of the heap of shot is a frequency curve of a commonly occurring type.

85. **The Probability of a Linear Function of Deviations.***—We shall now proceed to the analytical discussion of frequency distributions. We start from the fact that a measured quantity, such as the chest measure of a soldier, is liable to vary from one individual to another, or, in other words, to exhibit a *deviation* from its average value; and we shall suppose that this deviation is the total effect of a vast number of very small deviations, due to causes which operate independently of each other: denoting the small deviations by $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, we shall suppose that the total deviation is their sum, or, more generally, is a linear function of them, say, $\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \ldots + \lambda_n\epsilon_n$. We want to find the probability that in the case of a given individual soldier this total deviation shall have a value between (say) w_1 and w_2, where w_1 and w_2 are given numbers.

In the present section we shall not require to make use of the assumptions that the deviations $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are very small or very numerous.

Let $\int_a^b \phi_r(x)dx$ be the probability that the rth deviation ϵ_r has a value between a and b, so that $\phi_r(x)dx$ is the probability that it has a value between x and $x+dx$. Now the probability of the concurrence of any number of independent events is the product of the probabilities of the events happening separately; and hence the probability that the first constituent deviation has a value between ϵ_1 and $\epsilon_1+d\epsilon_1$, while the second has a value between ϵ_2 and $\epsilon_2+d\epsilon_2$, and so on, is

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* Laplace, *Théorie anal. des prob.*, livre II. chap. iv.; Poisson, *Connaissance des temps* (1827), p. 273.

$$\phi_1(\epsilon_1)\phi_2(\epsilon_2)\ \ldots\ \phi_n(\epsilon_n)d\epsilon_1 d\epsilon_2\ \ldots\ d\epsilon_n, \tag{1}$$

and the probability that $\lambda_1\epsilon_1+\ldots+\lambda_n\epsilon_n$ has a value between w_1 and w_2 is therefore the integral of the expression (1), taken over the field of integration for which $\lambda_1\epsilon_1+\lambda_2\epsilon_2+\ldots+\lambda_n\epsilon_n$ lies between w_1 and w_2.

Now by Fourier's Integral Theorem * the expression

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{w_1}^{w_2} e^{i\theta(\tau-x)}d\tau d\theta$$

has the value unity when x lies between w_1 and w_2, and has the value zero when x does not lie between these numbers. So instead of integrating the expression (1) over the field for which $w_1<\lambda_1\epsilon_1+\lambda_2\epsilon_2+\ldots+\lambda_n\epsilon_n<w_2$, we can first multiply the expression (1) by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{w_1}^{w_2} e^{i\theta(\tau-\lambda_1\epsilon_1-\lambda_2\epsilon_2-\ \ldots\ -\lambda_n\epsilon_n)}d\tau d\theta,$$

and then integrate the resulting expression over the field of all values of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.† The probability that $\lambda_1\epsilon_1+\lambda_2\epsilon_2+\ldots+\lambda_n\epsilon_n$ has a value between w_1 and w_2 is therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{w_1}^{w_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{i\theta(\tau-\lambda_1\epsilon_1-\lambda_2\epsilon_2-\ \ldots\ -\lambda_n\epsilon_n)}\phi_1(\epsilon_1)\phi_2(\epsilon_2)\ \ldots\ \phi_n(\epsilon_n)d\epsilon_1 d\epsilon_2\ \ldots\ d\epsilon_n d\tau d\theta.$$

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* The theorem in question is that under certain conditions (for which cf. Whittaker and Watson, *Modern Analysis*, § 9·7), the integral

$$\frac{1}{\pi}\int_0^{\infty} d\theta\int_{w_1}^{w_2}\cos\{\theta(x-\tau)\}f(\tau)d\tau$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\int_{w_1}^{w_2} e^{i\theta(x-\tau)}f(\tau)d\tau$$

has the value $f(x)$ when x lies between w_1 and w_2, and has the value zero when x does not lie between these numbers.

† This device is due to Cauchy, *Comptes Rendus*, 137 (1853), pp. 109, 264, 334.

Writing $\Omega_r(\theta)$ for $\int_{-\infty}^{\infty} e^{-i\theta x}\phi_r(x)dx$, the required probability is therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{w_1}^{w_2} e^{i\theta\tau}\Omega_1(\lambda_1\theta)\Omega_2(\lambda_2\theta)\ \ldots\ \Omega_n(\lambda_n\theta)d\tau d\theta,$$

or, performing the integration with respect to τ, the probability that $\lambda_1\epsilon_1+\lambda_2\epsilon_2+\ldots+\lambda_n\epsilon_n$ lies between $x-c$ and $x+c$ is

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\Omega(\theta)e^{i\theta x}\frac{\sin c\theta}{\theta}d\theta,$$

where $\Omega(\theta)$ is written for $\Omega_1(\lambda_1\theta)\Omega_2(\lambda_2\theta)\ \ldots\ \Omega_n(\lambda_n\theta)$; and therefore the probability that $\lambda_1\epsilon_1+\ldots+\lambda_n\epsilon_n$ lies between x and $x+dx$ is $\phi(x)dx$, where

$$\phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega(\theta)e^{i\theta x}d\theta. \tag{2}$$

From the last equation we have, by Fourier's Integral Theorem,

$$\Omega(\theta)=\int_{-\infty}^{\infty}e^{-i\theta x}\phi(x)dx, \tag{3}$$

so $\Omega(x)$ bears to $\phi(x)$ the same relation that $\Omega_r(x)$ bears to $\phi_r(x)$.

It appears, therefore, that when deviations are added to form an aggregate deviation, the corresponding functions $\Omega_r(\lambda_r\theta)$ are multiplied together to form the function $\Omega(\theta)$. The underlying reason for this is that when the Ω's are multiplied together, the arguments of the exponentials add together in precisely the same way as the deviations.

Now write s_k for $\int_{-\infty}^{\infty} x^k\phi_r(x)dx$, so that s_k is (in the language of the Theory of Probability) the *expectation* of the kth power of the rth deviation. Then by the definition of $\Omega_r(\theta)$ we have

$$\Omega_r(\theta)=1-i\theta s_1-\frac{\theta^2}{2!}s_2+\frac{i\theta^3}{3!}s_3+\frac{\theta^4}{4!}s_4+\ldots$$

Now let the logarithm of the series on the right, when expanded in ascending powers of θ, be denoted by

$$\log\Omega_r(\theta)=-i\theta p_1-\frac{\theta^2}{2!}p_2+\frac{i\theta^3}{3!}p_3+\frac{\theta^4}{4!}p_4-\ldots \tag{4}$$

so that

$$p_1=s_1,\quad p_2=-\begin{vmatrix} s_1 & 1 \\ s_2 & s_1\end{vmatrix},\quad p_3=\begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 1 \\ s_3 & s_2 & 2s_1\end{vmatrix}.$$

The quantity p_k is called * the *seminvariant of order k*. By (4) the fundamental formula

$$\Omega(\theta)=\Omega_1(\lambda_1\theta)\Omega_2(\lambda_2\theta)\ \ldots\ \Omega_n(\lambda_n\theta),$$

or $$\log\Omega(\theta)=\log\Omega_1(\lambda_1\theta)+\log\Omega_2(\lambda_2\theta)+\ \ldots\ +\log\Omega_n(\lambda_n\theta),$$

may now be expressed by the simple statement that *the kth seminvariant of a sum of deviations $\epsilon_1+\epsilon_2+\ldots+\epsilon_n$ is the sum of the kth seminvariants of the individual deviations*, with the further remark that the kth seminvariant of $\lambda\epsilon$ is λ^k times the seminvariant of ϵ.

86. **Approximation to the Frequency Function.**—We shall now derive from the result of the last section a formula of great practical importance by introducing the assumptions (which have not as yet been used) that the constituent deviations $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are very numerous and are individually very small. It is easily seen that by shifting the zero-point from which the deviation ϵ_r is measured we can secure the vanishing of its first seminvariant: we shall suppose this effected for each of the constituent deviations $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

Let $0, p_{21}, p_{31}, p_{41}, \ldots$ be the seminvariants for the deviation ϵ_1, let $0, p_{22}, p_{32}, p_{42}, \ldots$ be the seminvariants for the deviation ϵ_2, and so on; then, as we have seen, the seminvariants of the aggregate deviation ϵ are $P_1, P_2, \ldots$, where

$$\begin{aligned} P_1 &= 0, \\ P_2 &= \lambda_1^2 p_{21}+\lambda_2^2 p_{22}+\lambda_3^2 p_{23}+\ldots \\ P_3 &= \lambda_1^3 p_{31}+\lambda_2^3 p_{32}+\lambda_3^3 p_{33}+\ldots \end{aligned}$$

We shall suppose that P_2 is finite; and we shall further suppose that the constituent deviations are very numerous and are approximately of the same order of magnitude (or at any rate that a very large number of them are of the same order of magnitude and the rest are smaller than these), so the condition that P_2 is finite implies that each of the quantities $\lambda_r^2 p_{2r}$ is a finite quantity multiplied by $1/n$, and therefore $\lambda_r\sqrt{p_{2r}}$ is of the order of $n^{-\frac{1}{2}}$. This being so, $\lambda_r^3 p_{3r}$ will be of the order of $p_{3r}p_{2r}^{-\frac{3}{2}}n^{-\frac{3}{2}}$, and P_3 will be of the order of $p_{3r}p_{2r}^{-\frac{3}{2}}n^{-\frac{1}{2}}$, and so will—at any rate in a large class of cases—be small compared with P_2, on account of the factor $n^{-\frac{1}{2}}$. Similarly, P_4 will be small compared with P_3, and so on. Thus for the aggregate deviation we have

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* Thiele, *Almindelig Jagttagelseslaere*, Copenhagen, 1897: *Theory of Observations*, London (C. & E. Layton), 1903.

$$\log \Omega(\theta) = -\frac{\theta^2}{2!}P_2 + \frac{i\theta^3}{3!}P_3 + \frac{\theta^4}{4!}P_4 - \ldots,$$

where P_2, P_3, P_4, . . . are a rapidly decreasing sequence of quantities, and indeed in a large class of cases P_3, P_4, . . . are negligible compared with P_2. The probability that the aggregate deviation lies between x and $x+dx$ is therefore $\phi(x)dx$ where, by equation (2) of the last section,

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\theta x - \frac{\theta^2}{2!}P_2 + \frac{i\theta^3}{3!}P_3 + \frac{\theta^4}{4!}P_4 + \ldots}\, d\theta.$$

Writing

$$e^{\frac{i\theta^3}{3!}P_3 + \frac{\theta^4}{4!}P_4 + \ldots} = 1 + A(i\theta)^3 + B(i\theta)^4 + C(i\theta)^5 + \ldots,$$

where A, B, . . . are in a large class of cases negligible compared with P_2 (as follows from what has been said above), we have

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\theta x - \frac{1}{2}P_2\theta^2} . \{1 + A(i\theta)^3 + B(i\theta)^4 + C(i\theta)^5 + \ldots\}d\theta.$$

Now $$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\theta x - \frac{1}{2}P_2\theta^2} d\theta = \left(\frac{2}{\pi P_2}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2P_2}},$$

and therefore

$$\phi(x) = \left\{1 + A\left(\frac{d}{dx}\right)^3 + B\left(\frac{d}{dx}\right)^4 + C\left(\frac{d}{dx}\right)^5 + \ldots\right\}\frac{1}{(2\pi P_2)^{\frac{1}{2}}} e^{-\frac{x^2}{2P_2}} \quad \ldots \text{(A)}$$

In the large class of cases to which we have referred, when A, B, C, . . . are negligible, this becomes

$$\phi(x) = \frac{1}{(2\pi P_2)^{\frac{1}{2}}} e^{-\frac{x^2}{2P_2}} \quad \text{(B)}$$

The formula (B) is due to Laplace, and the more general form (A) to various later writers.* These formulae specify the nature of a frequency distribution in which the deviations from the average measure are due to the joint effect of a very large number of independent causes, each of which singly has only a very slight influence; the deviation of an individual's chest measure from the average, for instance, being thought of as affected by inheritance from a very large number of remote ancestors.

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* The first of the additional terms was found by Poisson, and the rest by J. P. Gram, *Om Raekkendviklinger* (1879); Thiele, *Forelaesninger over Almindelig Iagttagelseslaere* (1889); *Elementaer Iagttagelseslaere* (1897); Edgeworth, *Phil. Mag.* **41** (1896), p. 90; Bruns, *Ast. Nach.* **143** (1897) col. 329; Charlier, *Arkiv for Math.* **2** (1905), No. 20.

87. **Normal Frequency Distributions and Skew Frequency Distributions.**—Formula (B) of the last section shows that, at any rate in a large class of cases, when *a deviation* ϵ *is constituted by the summation of a very large number of independent deviations* $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, *the probability that* ϵ *lies between* x *and* $x + dx$ *is*

$$\frac{1}{(2\pi P_2)^{\frac{1}{2}}} e^{-\frac{x^2}{2P_2}} dx,$$

where P_2 *is independent of* x. This is called the *normal law of deviation.*

Now consider a frequency distribution consisting of N individuals (where N is a very large number), and suppose that the number of these individuals whose measure is between x and $x + dx$ is denoted by $N\phi(x)dx$, so that $\phi(x)$ is the probability that an individual taken at random in the whole group has a measure between x and $x + dx$. Let a denote the average value of x for all the individuals of the group, so that $(x - a)$ is the deviation of the measure of an individual from the average. Then on the assumption that the deviation is due to the operation of a very large number of independent causes, each of which makes only a very small contribution to the total deviation, we have seen that the probability of a total deviation between $x - a$ and $x - a + dx$ is (at any rate in a large class of cases)

$$\frac{1}{(2\pi P_2)^{\frac{1}{2}}} e^{-\frac{(x-a)^2}{2P_2}} dx,$$

where P_2 is independent of x. Frequency distributions to which this applies are called *normal frequency distributions.* Other distributions, for which the law (B) must be replaced by the more general law (A), are generally called *skew frequency distributions.*

As the subject of the present chapter is normal distributions, we shall say nothing more about skew distributions except that formula (A) leads to an expression for a skew frequency function as a series of the functions which are called *parabolic-cylinder functions* * or *Hermite's functions.* In most cases two or three terms of the series are adequate to represent the function with sufficient accuracy.

As an illustration of the way in which a great number of frequency distributions in nature conform to the normal law, we give the following data relating to 1000 observations made at Greenwich, of the Right Ascension of Polaris. Let x denote the deviation of one such measure from a value near the mean of all the measures, expressed as usual in seconds of time; let y denote the number of measures for which the amount of the deviation is x; let y' denote a value calculated from the theoretical equation of a normal frequency distribution,

$$y' = \frac{1000h}{\sqrt{\pi}} e^{-h^2(x-a)^2},$$

where $a = -0{\cdot}06$ and $h = 0{\cdot}6$. Then we have the following results:

x.	y.	y'.
−3·5	2	4
−3·0	12	10
−2·5	25	22
−2·0	43	46
−1·5	74	82
−1·0	126	121
−0·5	150	152
0	168	163
0·5	148	147
1·0	129	112
1·5	78	72
2·0	33	40
2·5	10	19
3·0	2	10

The agreement of y and y' is, generally speaking, good.

Ex.—*r measurements are made of a certain quantity, each measure being liable to an error which may have any value between ϵ and $-\epsilon$, all such values being equally likely. The sum of the r measures is denoted by s. Show that when ϵ tends to zero and r increases indefinitely in such a way that $\epsilon\sqrt{r}$ tends to a finite value k, then the probability that s lies between x and $x + dx$ tends to*

$$\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} \frac{dx}{k} e^{-\frac{3x^2}{2k^2}}.$$

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* For an account of these functions cf. Whittaker and Watson, *Modern Analysis*, § 16·7.

88. **The Reproductive Property of the Normal Law of Frequency.**—Consider a frequency distribution which is normal, so that if ϵ denotes the deviation of an individual measure from the average, then the probability that ϵ has a value between x and $x+dx$ is $\left(\text{writing } h^2 \text{ for } \frac{1}{2\text{P}_2}\right)$

$$\frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx.$$

The constant h was called by Gauss the *modulus of precision*.

The function $\Omega(\theta)$ is, by equation (3) of § 85,

$$\begin{aligned}\Omega(\theta) &= \int_{-\infty}^{\infty} e^{-i\theta x}\phi(x)dx \\ &= \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-i\theta x - h^2x^2}dx \\ &= e^{-\frac{\theta^2}{4h^2}},\end{aligned}$$

so

$$\log \Omega(\theta) = -\frac{1}{4h^2}\theta^2.$$

The only seminvariant is therefore of order two and has the value $\frac{1}{2h^2}$. From the additive property of seminvariants it follows that a deviation which is formed by the summation of any number of partial deviations, each of which obeys the normal law, has all its seminvariants zero except the second; and therefore this aggregate deviation itself obeys the normal law. This is the *reproductive property* or *group property* of the normal law of frequency; *an aggregate deviation, formed by the summation of any number of deviations which obey the normal law, itself obeys the normal law.*

89. **The Modulus of Precision of a Compound Deviation.**—Let a deviation ϵ be given as a linear combination $\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \ldots + \lambda_n\epsilon_n$ of a number of deviations $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$; each of which obeys the normal law, so that the probability that ϵ_r lies between x and $x + dx$ is

$$\frac{h_r}{\sqrt{\pi}}e^{-h_r^2x^2}dx,$$

where h_r is the modulus of precision for ϵ_r.

Then by the last section, the seminvariant for ϵ_r is $\frac{1}{2h_r^2}$, and therefore the seminvariant for $\lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n$, which is also of order two, is

$$\frac{1}{2}\left(\frac{\lambda_1^2}{h_1^2} + \frac{\lambda_2^2}{h_2^2} + \ldots + \frac{\lambda_n^2}{h_n^2}\right).$$

Now, as we have seen, ϵ also obeys the normal law of frequency; let the modulus of precision for ϵ be H, so the probability that ϵ lies between x and $x + dx$ is

$$\frac{\mathrm{H}}{\sqrt{\pi}} e^{-\mathrm{H}^2x^2} dx;$$

then the seminvariant for ϵ is $\frac{1}{2\mathrm{H}^2}$. Equating this to the value obtained above, we have

$$\frac{1}{\mathrm{H}^2} = \frac{\lambda_1^2}{h_1^2} + \frac{\lambda_2^2}{h_2^2} + \ldots + \frac{\lambda_n^2}{h_n^2}.$$

This important formula gives the modulus of precision of a compound deviation in terms of the moduli of precision of its constituent deviations, when these are of the normal type.

90. The Frequency Distribution of Tosses of a Coin.—The earliest example of a normal distribution of frequency was discovered by De Moivre in 1756 * in considering the following problem: *A coin is tossed* N *times, where* N *is a very large number* (which for convenience we suppose even and equal to $2n$): *to find the probability of exactly* $(\frac{1}{2}\mathrm{N} - p)$ *heads and* $(\frac{1}{2}\mathrm{N} + p)$ *tails.* The probability of exactly n heads and n tails is, by an elementary formula of probability, $\frac{1}{2^{2n}} \cdot \frac{2n!}{n!n!}$. Replacing the factorials by Stirling's approximate value (§ 70), namely, $z! = e^{-z}z^{z+\frac{1}{2}}(2\pi)^{\frac{1}{2}}$, this becomes $(\pi n)^{-\frac{1}{2}}$ or $\left(\frac{2}{\pi\mathrm{N}}\right)^{\frac{1}{2}}$. The probability of exactly $(n-p)$ heads and $(n+p)$ tails is $\frac{1}{2^{2n}} \frac{2n!}{(n-p)!(n+p)!}$; denoting the ratio of this to the former probability by E, we have therefore

$$\mathrm{E} = \frac{n!n!}{(n-p)!(n+p)!},$$

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* *Doctrine of Chances*, 3rd ed. (1756), p. 243.

so $$\log \mathrm{E} = 2 \log n! - \log (n-p)! - \log (n+p)!.$$

Replacing the factorials by Stirling's approximation, we have

$$\begin{aligned}
\log \mathrm{E} &= (2n+1) \log n - (n-p+\tfrac{1}{2}) \log (n-p) \\
&\qquad\qquad - (n+p+\tfrac{1}{2}) \log (n+p) \\
&= -\left\{\left(n-p+\frac{1}{2}\right)\log\left(1-\frac{p}{n}\right)+\left(n+p+\frac{1}{2}\right)\log\left(1+\frac{p}{n}\right)\right\} \\
&= \left(n-p+\frac{1}{2}\right)\left(\frac{p}{n}+\frac{p^2}{2n^2}+\ldots\right)+\left(n+p+\frac{1}{2}\right)\left(-\frac{p}{n}+\frac{p^2}{2n^2}+\ldots\right) \\
&= -\frac{p^2}{n}+\ldots, \\
&= -\frac{2p^2}{\mathrm{N}}+\ldots.
\end{aligned}$$

Combining these results, we see that *the probability of exactly* $(\frac{1}{2}\mathrm{N}-p)$ *heads and* $(\frac{1}{2}\mathrm{N}+p)$ *tails is approximately* $\left(\frac{2}{\pi \mathrm{N}}\right)^{\frac{1}{2}} e^{-\frac{2p^2}{\mathrm{N}}}$, whence *the probability that the number of heads will be between* $\frac{1}{2}\mathrm{N}+x\sqrt{\mathrm{N}}$ *and* $\frac{1}{2}\mathrm{N}+(x+dx)\sqrt{\mathrm{N}}$ *is approximately* $\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-2x^2}dx$. This is evidently a case of a normal frequency distribution.

Ex. 1.—*Show that the probability that the number of heads will be between* $\frac{1}{2}\mathrm{N}+\frac{1}{2}\sqrt{\mathrm{N}}$ *and* $\frac{1}{2}\mathrm{N}-\frac{1}{2}\sqrt{\mathrm{N}}$ *is* $\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{\sqrt{2}}} e^{-y^2}dy$ *or* 0·6827.

Ex. 2.—*A coin is tossed* 1000 *times. Show that an absolute majority of the* 2^{1000} *possible sequences gives the difference between the number of heads and number of tails less than twenty-two.*

91. An Illustration of the Non-Universality of the Normal Law.—The following example* shows that a great number of causes, each producing a very small deviation, do not always by their collective operation give rise to a frequency distribution of the normal type.

Suppose the law of frequency for each of the small constituent deviations $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ is $\phi(x) = \frac{1}{2}e^{-|x|}$, so

$$\phi(x) = \tfrac{1}{2}e^{-x} \text{ when } x \text{ is positive,}$$

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* Hausdorff, *Leipzig Ber.* **53** (1901), p. 152.

and $$\phi(x) = \tfrac{1}{2}e^x \text{ when } x \text{ is negative.}$$

In this case we have (§ 85) for each of the constituent deviations

$$\Omega_r(\theta) = \int_{-\infty}^{\infty} e^{-i\theta x}\phi_r(x)dx = \tfrac{1}{2}\int_0^{\infty} e^{-i\theta x - x}dx + \tfrac{1}{2}\int_{-\infty}^{0} e^{-i\theta x + x}dx$$
$$= \frac{1}{1+\theta^2}.$$

Let the compound deviation be defined to be ϵ, where

$$\epsilon = \frac{2}{\pi}\left(\epsilon_1 + \frac{1}{3}\epsilon_2 + \frac{1}{5}\epsilon_3 + \ldots\right).$$

Then (§ 85) the probability that ϵ has a value between x and $x + dx$ is $\phi(x)dx$, where

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(\theta)e^{i\theta x}d\theta,$$

and $$\Omega(\theta) = \Omega_1\left(\frac{2}{\pi}\theta\right)\Omega_2\left(\frac{2}{3\pi}\theta\right)\Omega_3\left(\frac{2}{5\pi}\theta\right)\ldots$$

$$= \frac{1}{1+\dfrac{2^2\theta^2}{\pi^2}}\cdot\frac{1}{1+\dfrac{2^2\theta^2}{3^2\pi^2}}\cdot\frac{1}{1+\dfrac{2^2\theta^2}{5^2\pi^2}}\ldots$$

$$= \frac{1}{\cosh\theta}.$$

Therefore $$\phi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\theta x}d\theta}{e^\theta + e^{-\theta}}.$$

This integral may readily be evaluated by integrating $\dfrac{e^{i\theta x}}{e^\theta + e^{-\theta}}$ in the plane of the complex variable θ round a rectangle of height π and indefinitely great breadth, one of whose sides is the real axis of θ. The result is

$$\phi(x) = \frac{1}{e^{\frac{1}{2}\pi x} + e^{-\frac{1}{2}\pi x}}.$$

Thus *the probability that the deviation* ϵ *has a value between* x *and* $x + dx$ *is* $\dfrac{1}{e^{\frac{1}{2}\pi x} + e^{-\frac{1}{2}\pi x}}dx$. This is clearly different from the normal law.

Theory asserts, and observation confirms the assertion, that the normal law is to be expected in a very great number of frequency distributions, but not in all. This state of things led in the past to much confusion of thought regarding the validity of the normal law, which was wittily referred to in Lippmann's remark to Poincaré:* "Everybody believes in the exponential law of errors: the experimenters, because they think it can be proved by mathematics; and the mathematicians, because they believe it has been established by observation."

92. **The Error Function.**—Having now established the theory of normal frequency distributions on a theoretical basis, we proceed to investigate the properties of these distributions. Denoting the modulus of precision by h, the probability that a deviation lies between x and $-x$ is

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-h^2x^2}h\,dx, \text{ or } \Phi(hx),$$

where

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.$$

On account of its importance in the Theory of Errors of Observation, $\Phi(x)$ is called the *Error Function.*

If the probability that a deviation lies between x and $-x$ is given, $\Phi(hx)$ is known, and therefore hx is determinate: so the deviation x corresponding to this probability decreases as h increases; that is to say, in the case when the deviations are the errors occurring in a set of measurements of a quantity, the precision of the measurements increases as h increases. This is the reason why h is called the *modulus of precision.*

We must now see how the function $\Phi(x)$ can be computed.†

1°. First we have

$$e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdot\cdot\cdot,$$

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* Poincaré, *Calcul des prob.* p. 149.

† For the analytical relations of the Error Function, cf. Whittaker and Watson, *Modern Analysis*, § 16.2.

and, integrating this,

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\left\{x-\frac{x^3}{1!3}+\frac{x^5}{2!5}-\frac{x^7}{3!7}+\ldots\right\}.$$

This series converges for all values of x.

For example, when $x=0{\cdot}5$, we have

$$\frac{x}{1}-\frac{x^3}{1!3}+\frac{x^5}{2!5}-\frac{x^7}{3!7}+\frac{x^9}{4!9}-\ldots$$

$$=\frac{1}{2}-\frac{1}{24}+\frac{1}{320}-\frac{1}{5376}+\frac{1}{110592}-\ldots$$

$$\begin{aligned}=0{\cdot}50000&-0{\cdot}04167\\ 0{\cdot}00313&\quad 0{\cdot}00019\\ 0{\cdot}00001&\end{aligned}$$

$$=0{\cdot}46128,$$

so
$$\begin{aligned}\Phi(x)&=0{\cdot}46128\times 1{\cdot}12838\\ &=0{\cdot}52050.\end{aligned}$$

2°. Next, write

$$\int_0^x e^{-x^2}dx=e^{-x^2}y.$$

Differentiating, we have

$$\frac{d}{dx}(e^{-x^2}y)=e^{-x^2},$$

or
$$\frac{dy}{dx}-2xy=1.$$

Now y evidently begins with a term x. So substitute $y=x+ax^3+bx^5+cx^7+\ldots$ in the last equation and equate coefficients of powers of x; therefore we obtain $3a-2=0$, $5b-2a=0$, $7c-2b=0$, etc., and therefore

$$y=x+\frac{2}{3}x^3+\frac{2^2}{3{\cdot}5}x^5+\frac{2^3}{3.5.7}x^7+\ldots$$

Therefore

$$\Phi(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}x\left\{1+\frac{1}{3}(2x^2)+\frac{1}{3.5}(2x^2)^2+\frac{1}{3.5.7}(2x^2)^3+\ldots\right\}.$$

This series converges for all values of x.

For example, when $x=0{\cdot}5$, we have

$$\Phi(x)=1{\cdot}12838\times e^{-0{\cdot}25}\times\frac{1}{2}$$

$$\times\left\{1+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3.5}\cdot\frac{1}{2^2}+\frac{1}{3.5.7}\cdot\frac{1}{2^3}+\frac{1}{3.5.7.9}\frac{1}{2^4}+\ldots\right\}$$

$$=1{\cdot}12838\times 0{\cdot}778801\times\frac{1}{2}\times\left\{1+\frac{1}{6}+\frac{1}{60}+\frac{1}{840}+\frac{1}{15120}+\ldots\right\}$$

$$\begin{aligned}=0{\cdot}56419\times 0{\cdot}778801\times{}&1{\cdot}000000\\&0{\cdot}166667\\&0{\cdot}016667\\&0{\cdot}001190\\&0{\cdot}000066\\&0{\cdot}000003\end{aligned}$$

$$=0{\cdot}56419\times 0{\cdot}778801\times 1{\cdot}184593$$

$$=0{\cdot}52050.$$

3⁰. Next, since $\sqrt{\pi}=2\int_0^\infty e^{-x^2}dx$, we have

$$\sqrt{\pi}-\sqrt{\pi}\Phi(x)=2\int_x^\infty e^{-x^2}dx$$

$$=\int_y^\infty e^{-y}y^{-\frac{1}{2}}dy,$$

where $x^2=y$,

$$=\left[-e^{-y}y^{-\frac{1}{2}}\right]_y^\infty-\frac{1}{2}\int_y^\infty e^{-y}y^{-\frac{3}{2}}dy$$

$$=e^{-y}y^{-\frac{1}{2}}+\frac{1}{2}\left[y^{-\frac{3}{2}}e^{-y}\right]_y^\infty+\frac{3}{2^2}\int_y^\infty e^{-y}y^{-\frac{5}{2}}dy$$

$$=e^{-y}y^{-\frac{1}{2}}-\frac{1}{2}e^{-y}y^{-\frac{3}{2}}+\frac{1.3}{2^2}e^{-y}y^{-\frac{5}{2}}-\ldots,$$

and therefore

$$\Phi(x)=1-\frac{e^{-x^2}}{x\sqrt{\pi}}\left\{1-\frac{1}{(2x^2)}+\frac{1.3}{(2x^2)^2}-\frac{1.3.5}{(2x^2)^3}+\ldots\right\}.$$

This is an *asymptotic expansion*,* which is convenient for computation when x is large.

A table of values † is

x.	$\Phi(x)$.	x.	$\Phi(x)$.	x.	$\Phi(x)$.	x.	$\Phi(x)$.
0·0	0·0000	0·6	0·6039	1·3	0·9340	2·0	0·9953
0·1	0·1125	0·7	0·6778	1·4	0·9523	2·1	0·9970
0·2	0·2227	0·8	0·7421	1·5	0·9661	2·2	0·9981
0·3	0·3286	0·9	0·7969	1·6	0·9763	2·3	0·9989
0·4	0·4284	1·0	0·8427	1·7	0·9838	2·4	0·9993
0·477	0·5001	1·1	0·8802	1·8	0·9891	2·5	0·9996
0·5	0·5205	1·2	0·9103	1·9	0·9928		

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* Cf. Whittaker and Watson, *Modern Analysis*, ch. viii.

† More extended tables will be found in a memoir by J. Burgess, *Trans. Roy. Soc. Edin.* **39** (1898), p. 257. Cf. also W. F. Sheppard, *Biometrika*, ii. (1903), p. 174.

Ex. 1.—*Show that the probability that the deviation is greater than x is slightly less than* 0·001 *when* $\Phi(hx)=0{\cdot}9981$ *or* $hx=2{\cdot}2$.

Ex. 2.—*Obtain the formula*

$$\Phi(x)=1-\frac{e^{-x^2}}{x\sqrt{\pi}}\cdot\cfrac{1}{1+\cfrac{y}{1+\cfrac{2y}{1+\cfrac{3y}{1+\ldots}}}},$$

where $y=\dfrac{1}{2x^2}$.

93. **Means connected with Normal Distributions.**—With the normal law of deviation

$$\frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx,$$

the arithmetic mean of the nth powers of the absolute values of the deviations is

$$\frac{2h}{\sqrt{\pi}}\int_0^\infty x^n e^{-h^2x^2}dx.$$

Writing $h^2x^2=y$, this becomes

$$\frac{1}{\sqrt{\pi}h^n}\int_0^\infty y^{\frac{n-1}{2}}e^{-y}dy$$

or *

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}h^n}.$$

In particular, writing σ for $\dfrac{1}{h\sqrt{2}}$, we have

Arithmetic mean of the absolute values of—

First powers of the deviations $=\dfrac{1}{h\sqrt{\pi}}=\sqrt{\dfrac{2}{\pi}}\cdot\sigma,$

Second " " " $=\dfrac{1}{2h^2}=\sigma^2,$

Third " " " $=\dfrac{1}{h^3\sqrt{\pi}}=\dfrac{2\sqrt{2}}{\sqrt{\pi}}\cdot\sigma^3,$

Fourth " " " $=\dfrac{3}{4h^4}=3\sigma^4,$

Fifth " " " $=\dfrac{2}{h^5\sqrt{\pi}}=\dfrac{8\sqrt{2}}{\sqrt{\pi}}\cdot\sigma^5.$

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* Cf. Whittaker and Watson, *Modern Analysis*, § 12.2.

From the above we see that *the square of the mean of the absolute deviations, divided by the mean of their squares, has the value* $\frac{2}{\pi}$.

Ex. 1.—*Show that in the curve* $y=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ *the product of the abscissa and the subtangent is constant.*

Ex. 2.—*Show that in the curve* $y=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ *the abscissae of the points of inflexion are* $\pm\sigma$.

94. Parameters connected with a Normal Frequency Distribution.

1°. Let the curve which represents graphically a normal frequency distribution be written

$$y=\frac{h}{\sqrt{\pi}}e^{-h^2(x-a)^2}.$$

Then the arithmetic mean of all the observations

$$=\int_{-\infty}^{\infty}\frac{h}{\sqrt{\pi}}e^{-h^2(x-a)^2}x\,dx$$

$$=a.$$

2°. The quantity σ which has been introduced in the last section is (as we have seen) the square root of the arithmetic mean of the squares of the deviations, and has the value

$$\sigma=\frac{1}{h\sqrt{2}}.$$

σ is called the *standard deviation*, or the *quadratic mean deviation* or the *error of mean square.*[*]

In terms of σ, the normal law is that the probability of a deviation between x and $x+dx$ is

$$\frac{1}{\sigma\sqrt{(2\pi)}}e^{-\frac{x^2}{2\sigma^2}}dx.$$

By the formula which gives the measure of precision of a linear function of deviations we see that the square of the standard deviation of a sum of quantities is the sum of the squares of the standard deviations of the separate quantities.

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[*] It was called by Gauss the *mean error.*

Hence *the standard deviation of the arithmetic mean of n quantities is* $n^{-\frac{1}{2}}\times$ *the standard deviation of one of the quantities.*

3°. The arithmetic mean of the absolute values of the deviations is called the *mean absolute deviation.** Denoting it by η, we have (as we have seen in § 93)

$$\eta=\frac{1}{h\sqrt{\pi}}=\sqrt{\frac{2}{\pi}}\cdot\sigma.$$

The arithmetic mean of all the observations that are greater than the mean a is $a+\sqrt{\frac{2}{\pi}}\cdot\sigma$, so, denoting this by A, we have

$$\sigma=\sqrt{\frac{\pi}{2}}(\mathrm{A}-a)=1\cdot253\,(\mathrm{A}-a).$$

This formula is convenient when the observations are given in the rough, not arranged in order of magnitude; for then, after adding them all and dividing by their number so as to obtain a, we have merely to add all that are greater than a and divide by their number in order to obtain $a+\sigma\sqrt{\frac{2}{\pi}}$.

4°. The *probable error* or *quartile* † is defined to be such that the chances are even whether the deviation exceeds it in absolute magnitude or is less than it. So if Q denotes the quartile, we have

$$\frac{1}{2}=\frac{2h}{\sqrt{\pi}}\int_0^{\mathrm{Q}}e^{-h^2\Delta^2}\,d\Delta=\frac{2}{\sqrt{\pi}}\int_0^{\mathrm{Q}h}e^{-t^2}\,dt,$$

which gives

$$\mathrm{Q}h=0\cdot476936=\rho\ \text{(say)},$$

or

$$\frac{\mathrm{Q}}{\sigma}=0\cdot67449\ \left(\text{roughly }\frac{2}{3}\right),$$

so

$$\sigma=1\cdot4826\mathrm{Q},\quad \mathrm{Q}=0\cdot67449\sigma.$$

Q is connected with η by the equations

$$\eta=1\cdot1829\mathrm{Q},\quad \mathrm{Q}=0\cdot84535\eta.$$

Q is evidently that deviation which stands in the middle of the sequence when the deviations are arranged in order of absolute magnitude. It therefore furnishes a means of determining the parameters h or σ of the curve as follows:

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* It was called by Laplace the *mean error*.

† The name is due to Galton.

Let there be n measurements. Take n equidistant points along a base-line, and at these points erect ordinates proportional respectively to the measures arranged in order of magnitude. We thus obtain a curve as in the diagram:

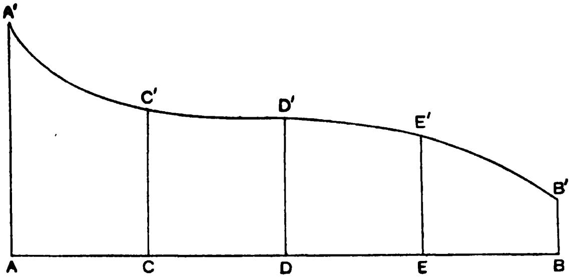

Fig. 16.

Divide the base AB into four equal parts by points of division C, D, E. The ordinate at D is the mean value a of the measures. The ordinate at E is such that there are as many measures greater than it as lie between it and a. The difference DD′ − EE′ or CC′ − DD′ is the Quartile Q.

More generally, we can determine the parameter h or σ of a curve by finding a measure x_1 such that (say) p per cent of all the measures fall within the interval between the mean a and x_1. In order to obtain σ, we have only to multiply $(x_1 - a)$ by a known numerical factor depending on p.

The degree of accuracy of this and the other methods of determining σ will be considered later (§ 103).

For expressing results in terms of probable error, the following form of the table of the function $\Phi(x)$ is useful:*

$0{\cdot}5000000 = \Phi(0{\cdot}4769363) = \Phi(\rho)$, where ρ/h is the probable error
$0{\cdot}6000000 = \Phi(0{\cdot}5951161) = \Phi(1{\cdot}247790\rho)$
$0{\cdot}7000000 = \Phi(0{\cdot}7328691) = \Phi(1{\cdot}536618\rho)$
$0{\cdot}8000000 = \Phi(0{\cdot}9061939) = \Phi(1{\cdot}900032\rho)$
$0{\cdot}8427008 = \Phi(1) \qquad\qquad = \Phi(2{\cdot}096716\rho)$
$0{\cdot}9000000 = \Phi(1{\cdot}1630872) = \Phi(2{\cdot}438664\rho)$
$0{\cdot}9900000 = \Phi(1{\cdot}8213864) = \Phi(3{\cdot}818930\rho)$
$0{\cdot}9990000 = \Phi(2{\cdot}3276754) = \Phi(4{\cdot}880475\rho)$
$0{\cdot}9999000 = \Phi(2{\cdot}7510654) = \Phi(5{\cdot}768204\rho)$
$1 \qquad\qquad = \Phi(\infty)$

---

* It is due to Gauss, *Werke*, **4**, p. 109.

We see, therefore,

The probability that the error exceeds—

2·438664 times the probable error is $\frac{1}{10}$,
3·818930 " " " $\frac{1}{100}$, etc.

Ex.—A number of bodies similar in shape and density differ slightly in size, their lengths being grouped about a mean a with standard deviation σ. If the weight of a body of length x is cx^3, show that the weights are grouped about a mean ca^3 with the standard deviation $3ca^2\sigma$.

95. **Determination of the Parameters of a Normal Frequency Distribution from a Finite Number of Observations.**—In the preceding section we have shown how to find the parameters a and σ (or η or h or Q) of a normal distribution, assuming tacitly that the observations are infinite in number so as to furnish continuous distribution. In reality, however, the number of observations is finite, and we have to determine the best values of the parameters from them.

Suppose there are n observations giving the values x_1, x_2, x_3, . . ., x_n respectively for x. The *a priori* probability of the value x_1 is

$$\frac{1}{\sqrt{(2\pi)}\sigma}e^{-\frac{(x_1-a)^2}{2\sigma^2}}$$

and therefore the *a priori* probability that the observations will give the set of values actually observed is

$$\left(\frac{1}{\sqrt{(2\pi)}\sigma}\right)^n e^{-\frac{(x_1-a)^2+(x_2-a)^2+\ .\ .\ .+(x_n-a)^2}{2\sigma^2}}$$

The most probable hypothesis regarding a and σ is that which makes this quantity a maximum when x_1, x_2, . . ., x_n are supposed given. Taking logs, we see that

$$n\log\sigma+\frac{(x_1-a)^2+(x_2-a)^2+\ .\ .\ .+(x_n-a)^2}{2\sigma^2},$$

or Π say, must be a minimum, and therefore $\frac{\partial\Pi}{\partial a}=0$, which gives

$$0=(x_1-a)+(x_2-a)+\ .\ .\ .+(x_n-a),$$

or $$a=\frac{1}{n}(x_1+x_2+\ .\ .\ .+x_n). \tag{1}$$

Moreover, $\frac{\partial \Pi}{\partial \sigma}=0$, which gives

$$0=\frac{n}{\sigma}-\frac{(x_1-a)^2+(x_2-a)^2+\ldots+(x_n-a)^2}{\sigma^3},$$

and therefore

$$\sigma^2=\frac{(x_1-a)^2+(x_2-a)^2+\ldots+(x_n-a)^2}{n}. \qquad (2)$$

Thus the formulae (1) and (2) for a and σ are determined directly from the theory of Inductive Probability. This is, strictly speaking, the only correct method when the number of observations is *finite*.

Ex. 1.—*Numbers which are given as decimals to some definite number of places are usually* forced, *i.e. the last digit retained is increased by unity when the first digit not retained is* 5, 6, 7, 8, *or* 9. *Find the standard deviation or mean error to which a number is liable, due to abbreviation with forcing.*

Denoting the standard deviation by σ, we have

$$\sigma^2=\frac{\text{Sum of squares of all possible omitted ``tails''}}{\text{Number of these ``tails''}}$$

$$=\frac{\int_{-0\cdot5}^{0\cdot5}\epsilon^2 d\epsilon}{\int_{-0\cdot5}^{0\cdot5}d\epsilon},$$ since all tails ϵ between $-0\cdot5$ and $0\cdot5$ (in units of the last digit retained) are equally probable,

$$=\frac{(0\cdot5)^3-(-0\cdot5)^3}{3},$$

giving $$\sigma=0\cdot2887.$$

Ex. 2.—Hence show that the mean error liable to occur in the sum of 1000 numbers, each of which has been abbreviated with forcing, is less than 10 units of the last place.

Ex. 3.—*A quantity is repeatedly measured, the measures being subject to errors of observation. Assuming the law of facility of error to be such that the probability of a value between x and $x+dx$ is*

$$\tfrac{1}{2}k^2e^{-k^2|x-b|}dx,$$

where b and k are constants, determine the most probable values of the parameters b and k when n observations give the values $x_1, x_2, \ldots, x_n$, for x; the values $x_1, x_2, \ldots$ being arranged in ascending order of magnitude.

Here the *a priori* probability that the observations will give this set of values is proportional to

$$\frac{1}{2^n}k^{2n}e^{-k^2\{|x_1-b|+|x_2-b|+\ldots+|x_n-b|\}}$$

The most probable hypothesis regarding b and k is that which makes this quantity a maximum, when $x_1, x_2, \ldots, x_n$ are supposed given.

Taking logs, we see that

$$\Pi \equiv 2n \log k - k^2\{\,|x_1 - b| + |x_2 - b| + \ldots + |x_n - b|\,\}$$

is a maximum, and therefore $\frac{\partial \Pi}{\partial b} = 0$. Suppose that b lies between x_r and x_{r+1}. Then

$$\frac{\partial}{\partial b}\{\,|x_1 - b| + |x_2 - b| + \ldots + |x_n - b|\,\}$$

$$= \frac{\partial}{\partial b}\{(b - x_1) + (b - x_2) + \ldots + (b - x_r) + (x_{r+1} - b) + \ldots + (x_n - b)\}$$

$$= (1 + 1 + \ldots + 1)_{r \text{ terms}} + (-1 - 1 \ldots - 1)_{(n-r) \text{ terms}},$$

and therefore $\frac{\partial \Pi}{\partial b}$ is zero when $r = n - r$ or $r = \frac{1}{2}n$. We see, therefore, that *the most plausible value of b is that one of the quantities $x_1, x_2, \ldots, x_n$ which stands in the middle of the sequence when they are arranged in order of magnitude.* This value is called the *median.* The condition $\frac{\partial \Pi}{\partial k} = 0$ then gives for the determination of the most probable value of k the equation

$$\frac{1}{k^2} = \frac{1}{n}\{\,|x_1 - b| + |x_2 - b| + \ldots + |x_n - b|\,\}.$$

96. The Practical Computation of a and σ.

—In calculating a and σ, when we are given that the measures $x_1, x_2, x_3, \ldots$ have occurred $y_1, y_2, y_3, \ldots$ times respectively, we generally find it convenient to subtract some fixed number c from each of the x's in order to have smaller numbers to deal with. Write $x_1 - c = \xi_1$, $x_2 - c = \xi_2$, etc.

Then we form a column of the quantities $\xi_1, \xi_2, \xi_3, \ldots$

then „ „ „ $y_1\xi_1, y_2\xi_2, y_3\xi_3, \ldots$

then „ „ „ $y_1\xi_1^2, y_2\xi_2^2, y_3\xi_3^2, \ldots$

the value of $y_r\xi_r^2$ being obtained by multiplying $y_r\xi_r$ by ξ_r, and sum the column of y's and the last two columns. We then have

$$a = \frac{\Sigma xy}{\Sigma y} = \frac{\Sigma(c + \xi)y}{\Sigma y} = c + \frac{\Sigma y\xi}{\Sigma y}, \qquad (1)$$

and
$$\sigma^2=\frac{\Sigma y(x-a)^2}{\Sigma y}=\frac{\Sigma y(c+\xi-a)^2}{\Sigma y}$$

$$=\frac{\Sigma y\xi^2}{\Sigma y}+2(c-a)\frac{\Sigma y\xi}{\Sigma y}+(c-a)^2$$

$$=\frac{\Sigma y\xi^2}{\Sigma y}-2(c-a)^2+(c-a)^2,$$

or
$$\sigma^2=\frac{\Sigma y\xi^2}{\Sigma y}-(c-a)^2. \tag{2}$$

Thus a and σ are determined.

As a control for the computation of a, we may use the equation

Sum of all positive residuals = Sum of all negative residuals,

and as a control for the computation of σ, we may use the equation

$$n\sigma^2=\Sigma(x-a)^2$$

or
$$n\sigma^2=\Sigma M^2-2a\Sigma M+a\Sigma M=\Sigma M^2-a\Sigma M,$$

where ΣM denotes the sum of the measures and ΣM^2 denotes the sum of the squares of the measures.

97. Examples of the Computation of a and σ.

Ex. 1.—*The chest measurements of* 10,000 *men are given below, x being the measure in inches and y being the number of men who have these measures. Find the two constants a and σ which specify the frequency curve, obtaining the standard deviation firstly from the mean-square of the deviations, and secondly from the mean absolute deviation.*

x.	y.	$\xi(=x-40)$.	$y\xi$.	$y\xi^2$.
33	6	−7	− 42	294
34	35	−6	− 210	1260
35	125	−5	− 625	3125
36	338	−4	−1352	5408
37	740	−3	−2220	6660
38	1303	−2	−2606	5212
39	1810	−1	−1810	1810
40	1940	0	0	0
41	1640	1	1640	1640
42	1120	2	2240	4480
43	600	3	1800	5400
44	222	4	888	3552
45	84	5	420	2100
46	30	6	180	1080
47	5	7	35	245
48	2	8	16	128
Total	10000		−1646	42394

Therefore $$a = 40 - \frac{1646}{10000} = 39{\cdot}835,$$

and $$\sigma^2 = \frac{42394}{10000} - (0{\cdot}1646)^2 = 4{\cdot}2123,$$

so $$\sigma = 2{\cdot}052.$$

In order to find σ from the mean absolute deviation, we have the following table:

x.	y.	$\xi(=x-40)$.	$y\xi$.
33	6	-7	-42
34	35	-6	-210
35	125	-5	-625
36	338	-4	-1352
37	740	-3	-2220
38	1303	-2	-2606
39	1810	-1	-1810
39·668 (= 39·500 to 39·835)	650	$-0{\cdot}332$	-216
Total	5007		-9081

Now

$$a - \sigma\sqrt{\frac{2}{\pi}} = \text{Arithmetic mean of all observations less than } a$$

$$= \frac{\Sigma yx}{\Sigma y} = 40 + \frac{\Sigma y\xi}{\Sigma y}$$

or $$39{\cdot}835 - \sigma\sqrt{\frac{2}{\pi}} = 40 - \frac{9081}{5007}.$$

Therefore $$\sigma\sqrt{\frac{2}{\pi}} = 1{\cdot}649,$$

so $$\sigma = 1{\cdot}649 \times 1{\cdot}253$$

or $$\sigma = 2{\cdot}066.$$

Ex. 2.—*Find the mean value and standard deviation for the following frequency distribution, y being the number of occurrences of the measure x. Trace the normal curve which has this mean value and standard deviation and compare it with the original data.*

x.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
y.	6	10	2	11	15	18	18	34	32	39	39	45	50	43

x.	15	16	17	18	19	20	21	22	23	24	25.
y.	38	26	34	28	22	17	11	12	11	4	4.

Ex. 3.—*Determine the standard deviation from the mean absolute deviation in this last example.*

Ex. 4.—*The chest measures of* 10,000 *men are as follows, x denoting a measure in inches, and y the number of men who have that measure. Determine the standard deviation by finding the place of the quartile.*

x.	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48.
y.	5	31	141	322	732	1305	1867	1882	1628	1148	645	160	87	38	7	2.
Σy.	5	36	177	499	1231	2536	4403	6285	7913	9061	9706	9866	9953	9991	9998	10000.

It is to be remembered that under "$x = 38$" are grouped all men whose measures are from $37\frac{1}{2}$ to $38\frac{1}{2}$ inches. The sum of all measures from the lowest to $x = 38$ (*i.e.* to $38\frac{1}{2}$ inches) is 2536; and the rate of increase of men is about 16 per $\frac{1}{100}$ inch at $38\frac{1}{2}$, so that 2500 will be the sum of all to 38·478 inches.

The Arithmetic mean a of all the measures is found to be 39·834, and

$$Q = 39{\cdot}834 - 38{\cdot}478 = 1{\cdot}356,$$

so

$$\sigma = 1{\cdot}483 \times Q = 1{\cdot}483 \times 1{\cdot}356 = 2{\cdot}01.$$

Let us now find the quartile from the other end of the sequence. The sum of all measures from the largest down to $x = 42$ (*i.e.* to $41\frac{1}{2}$ inches) is 2087. Adding half the number opposite 41 in the table, we see that the number down to 41 inches is 2901, which is 2500 + 401. The quartile will therefore be at $41 + x$, where $x = \frac{401}{1628} = 0{\cdot}246$,

so that

$$Q = 41{\cdot}246 - 39{\cdot}834 = 1{\cdot}412,$$

and

$$\sigma = 1{\cdot}483 \times 1{\cdot}412 = 2{\cdot}09.$$

The discordance between this and the former value is, of course, due to the absence of perfect normality and continuity in the distribution.

The mean of the two values of σ obtained by the quartile method is therefore $\sigma = 2{\cdot}05$. This is close to the values obtained by the method of § 97 and the mean absolute deviation method, which are found to be 2·05 and 2·06 respectively.

98. Computation of Moments by Summation.—The quantities $M_0 = \Sigma y$, $M_1 = \Sigma xy$, $M_2 = \Sigma x^2 y$, . . ., which are called the *moments*, may be readily formed by mere addition in the following way.

We use the notation Σy (read "sum of y") to denote the function whose first difference is the function y, so that if $\Delta u = y$, then $u = \Sigma y$. The symbol Σ corresponds, in the Calculus of Differences, to the symbol $\int$ which represents indefinite integration in the Infinitesimal Calculus. Just as a column of first differences of u can be formed from a column of values of u by subtractions, so a column of values of Σy can be formed from a column of values of y by additions.

Suppose the set of given values of y is $y_r, y_{r+1}, y_{r+2}, \ldots, y_{r+n}$, corresponding respectively to the values $r, (r+1), (r+2), \ldots, (r+n)$, of x. Form a table of sums of the function y by additions from the foot of the column, thus:

x.	y.	Σy.	$\Sigma^2 y$.
r	y_r	$y_r + y_{r+1} + \ldots + y_{r+n}$	$\ldots$
$r+1$	y_{r+1}	$y_{r+1} + \ldots + y_{r+n}$	$y_{r+1} + 2y_{r+2} + \ldots + ny_{r+n}$
$r+2$	y_{r+2}	$y_{r+2} + \ldots + y_{r+n}$	$y_{r+2} + 2y_{r+3} + \ldots + (n-1)y_{r+n}$
$\vdots$			
$r+n-2$	y_{r+n-2}	$y_{r+n-2} + \ldots + y_{r+n}$	$y_{r+n-2} + 2y_{r+n-1} + 3y_{+rn}$
$r+n-1$	y_{r+n-1}	$y_{r+n-1} + y_{r+n}$	$y_{r+n-1} + 2y_{r+n}$
$r+n$	y_{r+n}	y_{r+n}	y_{r+n}
			Sum = S_2

The uppermost number on the Σy column is evidently M_0, *the moment of zero order*. Let the uppermost number in the Σy column be denoted by S_0, and the uppermost number in the $\Sigma^2 y$ column be denoted by S_1. Then

$$\begin{aligned}S_0 &= M_0,\\ S_1 &= y_{r+1} + 2y_{r+2} + 3y_{r+3} + \ldots + ny_{r+n}\\ &= M_1 - rM_0,\end{aligned}$$

where $M_1 = ry_r + (r+1)\,y_{r+1} + \ldots + (r+n)\,y_{r+n}$ is the *moment of order* 1.

The sum S_2 of the $\Sigma^2 y$ column is

$$\begin{aligned}S_2 &= y_{r+1} + 3y_{r+2} + 6y_{r+3} + \ldots + \frac{n(n+1)}{2}y_{r+n}\\ &= \tfrac{1}{2}\{M_2 - (2r-1)\,M_1 + r\,(r-1)\,M_0\},\end{aligned}$$

where $M_2 = r^2 y_r + (r+1)^2 y_{r+1} + \ldots + (r+n)^2 y_{r+n}$ is *the moment of order* 2. Thus M_1 and M_2 may be determined from the equations

$$\begin{aligned}M_1 &= S_1 + rM_0,\\ M_2 &= 2S_2 + (2r-1)\,S_1 + r^2 M_0,\end{aligned}$$

and therefore if these relate to a frequency distribution of mean a and standard deviation σ, we have

$$a=\frac{M_1}{M_0}=\frac{S_1}{M_0}+r,$$

$$\sigma^2=\frac{M_2}{M_0}-\frac{M_1^2}{M_0^2}=\frac{2S_2}{M_0}-\frac{S_1}{M_0}-\left(\frac{S_1}{M_0}\right)^2,$$

or

$$\sigma^2=\frac{2S_2}{S_0}-\frac{S_1}{S_0}-\left(\frac{S_1}{S_0}\right)^2.$$

The higher moments may then be found in the same way if desired.*

Let us apply this to the computation of a and σ for the chest measures of Ex. 1, § 97.

x.	y.	Σy.	$\Sigma^2 y$.
33	6	10000 ($=S_0$)	...
34	35	9994	68354 ($=S_1$)
35	125	9959	58360
36	338	9834	48401
37	740	9496	38567
38	1303	8756	29071
39	1810	7453	20315
40	1940	5643	12862
41	1640	3703	7219
42	1120	2063	3516
43	600	943	1453
44	222	343	510
45	84	121	167
46	30	37	46
47	5	7	9
48	2	2	2
			$S_2=288852$

Therefore

$$M_0=10000=S_0,$$

$$a=\frac{S_1}{S_0}+r=6{\cdot}8354+33=39{\cdot}8354,$$

and

$$\sigma^2=2\frac{S_2}{S_0}-\frac{S_1}{S_0}-\left(\frac{S_1}{S_0}\right)^2=57{\cdot}7704-6{\cdot}8354-(6{\cdot}8354)^2$$

$$=4{\cdot}212 \text{ as before, giving } \sigma=2{\cdot}052.$$

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* For a fuller investigation, showing the advantages of *central* or *mean* sums, cf. a note contributed by G. J. Lidstone to G. F. Hardy's *Construction of Tables of Mortality* (London, 1909), printed on pp. 124-128 of that work.

Ex. 1.—*Using the data of Ex.* 2, § 97, *compute the value of* σ *by the method of summation.*

Ex. 2.—*The mean daily temperature at Brussels on the* 310 *days of the months of July in ten years was as follows:*

Mean temperature	11°·5	12°·5	13°·5	14°·5	15°·5	16°·5	17°·5	18°·5
Number of days	1	9	21	24	33	32	49	35
Mean temperature	19°·5	20°·5	21°·5	22°·5	23°·5	24°·5	25°·5	26°·5
Number of days	31	24	21	17	7	2	3	1

Find the mean temperature and standard deviation by the method of summation.

99. **Sheppard's Corrections.**—We shall now investigate a correction which is to be applied in calculations such as those of the preceding section.

Let $y = f(x)$ be the equation of a frequency curve. The integral

$$\int_{-\infty}^{\infty} x^p f(x)\,dx,$$

which is called the *p*th *moment* of the curve, will be denoted by m_p.

The statistical data which specify frequency curves are often given in a summarised form from which the moments cannot be computed directly with accuracy. For instance, in statistics of the chest measurements of a group of men, all men whose chest measure lies between $38\frac{1}{2}$ and $39\frac{1}{2}$ inches might be given under the heading "39 inches"; all men with measures between $39\frac{1}{2}$ and 40 inches might be given under the heading "40 inches," and so on. The number given under the heading "40 inches" is therefore not a true *ordinate* of the frequency curve, but is really the area of that strip of the frequency curve which is comprised between the ordinates at $x = 39\frac{1}{2}$ and $x = 40\frac{1}{2}$; that is, it is

$$\int_{39\frac{1}{2}}^{40\frac{1}{2}} f(x)\,dx.$$

Suppose, then, that $\ldots x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ are the values of x for which statistical data are given, these values being spaced at equal intervals w, and suppose that the statistical data are the numbers $\ldots u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$, where $u_s = \int_{x_s - \frac{1}{2}w}^{x_s + \frac{1}{2}w} f(x)\,dx$, and suppose we calculate the quantities

$$m_p' = \sum_{s=-\infty}^{\infty} x_s^p u_s, \tag{1}$$

so that m_p' is a rough value for the *p*th moment, obtained by a process which is equivalent to collecting at x_s all the individual measures between $x_s - \frac{1}{2}w$ and $x_s + \frac{1}{2}w$.

The problem before us is *to obtain a formula which will enable us to calculate the true moments* m_p *from the rough moments* m_p'. We shall suppose that the frequency curve has close contact with the axis of x at both ends; so that

$$m_p = \int_{-\infty}^{\infty} x^p f(x)dx = w \sum_{s=-\infty}^{\infty} x_s^p f(x_s). \tag{2}$$

Now, by the Newton-Stirling formula (§ 23),

$$f(x_s + nw) = f(x_s) + n\tfrac{1}{2}\{\Delta f(x_s) + \Delta f(x_s - w)\} + \frac{n^2}{2!}\Delta^2 f(x_s - w)$$

$$+ \frac{n(n^2-1)}{3!}\tfrac{1}{2}\{\Delta^3 f(x_s - w) + \Delta^3 f(x_s - 2w)\} + \frac{n^2(n^2-1)}{4!}\Delta^4 f(x_s - 2w) + \dots,$$

so we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_s + nw)dn = f(x_s) + \frac{1}{3!\,2^2}\Delta^2 f(x_s - w) - \frac{17}{3.5!\,2^4}\Delta^4 f(x_s - 2w) + \dots$$

In particular, when $f(x) = e^x$ we have

$$\Delta^{2n} f(x_s - nw) = 2^{2n} e^{x_s} \sinh^{2n} \tfrac{1}{2}w.$$

Putting $\theta = \frac{1}{2}w$ and dividing throughout by e^{x_s}, the expansion reduces to

$$\frac{\sinh\theta}{\theta} = 1 + \frac{1}{3!}\sinh^2\theta - \frac{17}{3.5!}\sinh^4\theta + \dots,$$

whence the coefficients in the general expansion may be determined readily.

Since
$$\frac{1}{w}\int_{x_s - \frac{1}{2}w}^{x_s + \frac{1}{2}w} f(x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_s + nw)dn,$$

we may write
$$\frac{1}{w}u_s = \frac{\sinh\theta}{\theta} f(x_s),$$

where $\sinh^{2n}\theta$ stands for the operation $(\frac{1}{4}\Delta^2 E^{-1})^n$. We have therefore from (1)

$$\frac{1}{w}m_p' = \sum_{s=-\infty}^{\infty} \frac{1}{w} x_s^p u_s = \sum_{s=-\infty}^{\infty} x_s^p \frac{\sinh\theta}{\theta} f(x_s).$$

Also since
$$\sum_{s=-\infty}^{\infty} x_s^p f(x_s + qw) = \sum_{s=-\infty}^{\infty} f(x_s) x_{s-q}^p,$$

the terms of both sides of this equality being the same but counted differently, we have

$$\sum_{s=-\infty}^{\infty} x_s^p (\Delta^2 E^{-1})^n f(x_s) = \sum_{s=-\infty}^{\infty} f(x_s)(\Delta^2 E^{-1})^n x_s^p.$$

Therefore

$$\frac{1}{w}m_p' = \sum_{s=-\infty}^{\infty} f(x_s)\frac{\sinh\theta}{\theta} x_s^p = \sum_{s=-\infty}^{\infty} f(x_s)\left\{1 + \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \frac{\theta^6}{7!} + \dots\right\} x_s^p.$$

Now if D denotes $\frac{d}{dx}$, we have

$$\mathrm{E} = e^{w\mathrm{D}} \text{ and } \Delta = e^{w\mathrm{D}} - 1,$$

so

$$\tfrac{1}{4}\Delta^2\mathrm{E}^{-1} = \sinh^2 \frac{w\mathrm{D}}{2}.$$

Hence we see that $\theta \equiv \frac{1}{2}w\mathrm{D}$ and

$$\begin{aligned} m_p' &= w \sum_{s=-\infty}^{\infty} f(x_s)\left\{1 + \frac{w^2\mathrm{D}^2}{3\,!\,2^2} + \frac{w^4\mathrm{D}^4}{5\,!\,2^4} + \frac{w^6\mathrm{D}^6}{7\,!\,2^6} + \ldots\right\}x_s^p \\ &= w \sum_{s=-\infty}^{\infty} f(x_s)\left\{x_s^p + \frac{w^2}{3\,!\,2^2}p(p-1)x_s^{p-2}\right. \\ &\qquad\qquad \left. + \frac{w^4}{5\,!\,2^4}p(p-1)(p-2)(p-3)x_s^{p-4} + \ldots\right\}, \end{aligned}$$

or finally, substituting from equation (2), we have

$$m_p' = m_p + \frac{w^2}{3\,!\,2^2}p(p-1)m_{p-2} + \frac{w^4}{5\,!\,2^4}p(p-1)(p-2)(p-3)m_{p-4} + \ldots \quad (3)$$

Taking $p = 1, 2, 3, 4, 5$ in succession in (3), we have

$$\begin{aligned} m_1' &= m_1, \\ m_2' &= m_2 + \tfrac{1}{12}w^2m_0, \\ m_3' &= m_3 + \tfrac{1}{4}w^2m_1, \\ m_4' &= m_4 + \tfrac{1}{2}w^2m_2 + \tfrac{1}{80}w^4m_0, \\ m_5' &= m_5 + \tfrac{5}{6}w^2m_3 + \tfrac{1}{16}w^4m_1, \end{aligned}$$

and hence

$$\begin{aligned} m_0 &= m_0', \\ m_1 &= m_1', \\ m_2 &= m_2' - \tfrac{1}{12}w^2m_0', \\ m_3 &= m_3' - \tfrac{1}{4}w^2m_1', \\ m_4 &= m_4' - \tfrac{1}{2}w^2m_2' + \tfrac{7}{240}w^4m_0', \\ m_5 &= m_5' - \tfrac{5}{6}w^2m_3' + \tfrac{7}{48}w^4m_1'. \end{aligned}$$

These formulae, which express the true moments in terms of the approximate moments, the curve being supposed to have close contact with the axis of x at both ends, are due to W. F. Sheppard.*

100. **On Fitting a Normal Curve to an Incomplete Set of Data.**—It sometimes happens that we wish to determine a normal curve when we know only the ordinates $y_1, y_2, y_3, \ldots$ of a set of points of abscissa $x_1, x_2, x_3, \ldots$ which are extended over part of the curve, no information being available regarding the rest of the curve. In this case it is best to treat the problem as one of fitting a parabolic curve $z = a + bx + cx^2$ for the given values of z where $z = \log y$. The constants a, b, c may be determined by the method of Least Squares (Chapter IX.).

101. **The Probable Error of the Arithmetic Mean.**—Let

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* *Proc. Lond. Math. Soc.* **29** (1898), p. 353.

m_0 denote the arithmetic mean of n measures $M_1, M_2, \ldots, M_n$, so that

$$m_0 = \frac{1}{n}(M_1 + M_2 + \ldots + M_n).$$

Then if h_1 denote the modulus of precision of the arithmetic mean and h denote the modulus of precision of a single observation, from the formula for the precision of a linear function (§ 89), we have

$$\frac{1}{h_1^2} = n \cdot \frac{\left(\frac{1}{n}\right)^2}{h^2}$$

or

$$h_1 = h\sqrt{n}.$$

The probable error of a single measure being connected with the modulus of precision by the equation $Q = \frac{\rho}{h}$, where $\rho = 0{\cdot}476936$ (§ 94), we see at once that *the probable error of the arithmetic mean is* $\frac{1}{\sqrt{n}}$ *times the probable error of a single observation.*

102. **The Probable Error of the Median.**—Instead of taking the arithmetic mean of the measures of an observed quantity as the estimate of the true value of the quantity, suppose we now arrange all the n measures in the order of their magnitude and select the middle one, which is called the *median.** Usually the median will be close to the arithmetic mean and will furnish an independent estimate of the observed quantity.

A more precise definition of the median is as follows: †

Let $a_1, a_2, a_3, \ldots, a_n$ be a set of real numbers, which may or may not be all distinct. Let

$$S_2(x) = \sum_{i=1}^{n} (x - a_i)^2.$$

The value of x which reduces $S_2(x)$ to a minimum is the *arithmetic mean* of the numbers $a_1, \ldots, a_n$. If the condition that $S_2(x)$ be a minimum is replaced by the condition that

$$S_1(x) = \sum_{i=1}^{n} |x - a_i|$$

be reduced to a minimum, the *median* of the a's is obtained. It is uniquely defined whenever n is odd; if the numbers a_i are arranged in order of magnitude, so that

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* Cf. § 95, Ex. 2.

† Dunham Jackson, *Bull. Am. Math. Soc.* **27** (1921), p. 160.

$$a_1 \leqq a_2 \leqq . \; . \; . \leqq a_n,$$

and if $n=2k-1$, the median is simply a_k, the middle one of the a's. The median is uniquely defined also when n is even, $n=2k$, if it happens that $a_k=a_{k+1}$, being then equal to this common value. Otherwise the definition is satisfied by any number x belonging to the interval

$$a_k \leqq x \leqq a_{k+1},$$

and the median is to this extent indeterminate. But for each value of $p>1$ there is a definite number $x=x_p$ which minimises the sum

$$S_p(x)=\sum_{i=1}^{n} |x-a_i|^p,$$

and x_p approaches a definite limit X as p approaches 1. The value of X coincides with the median as already defined, in the cases where that definition is determinate, and when $n=2k$ and $a_k \neq a_{k+1}$, X is a definite number between a_k and a_{k+1}. It thus serves to supplement the former definition.

We shall now find the probable error of the median in order that we may judge of the relative advantages of the arithmetic mean and the median as estimates of the true value of the observed quantity.

Suppose we have n measures (where n is supposed a great number). Then the probability that any one measure exceeds the true value is $\frac{1}{2}$ and by § 90 the probability that exactly $(\frac{1}{2}n+r)$ of the measures exceed the true value is

$$\sqrt{\left(\frac{2}{\pi n}\right)}e^{-\frac{2r^2}{n}}.$$

Now if h is the modulus of precision of the measures, the probability that a measure lies at a distance between 0 and ξ from the true value where ξ is small is $\int_0^\xi \frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx$ or approximately $\frac{h}{\sqrt{\pi}}\xi$; and therefore of n measures the number between 0 and ξ is $\frac{nh}{\sqrt{\pi}}\xi$. If this number is r, we have $r=\frac{nh}{\sqrt{\pi}}\xi$. Of the $(\frac{1}{2}n+r)$ measures greater than the true value, $\frac{1}{2}n$ exceed ξ. Therefore the probability that the median is at the point ξ is

$$\sqrt{\left(\frac{2}{\pi n}\right)}e^{-\frac{2nh^2\xi^2}{\pi}}$$

Denoting the value of ξ corresponding to $(r+1)$ by $\xi + d\xi$, then $r+1 = \frac{nh(\xi + d\xi)}{\sqrt{\pi}}$, and the change from r to $r+1$ corresponds to an increase of $\frac{nh}{\sqrt{\pi}} d\xi$ in the number of measures, so $d\xi = \frac{\sqrt{\pi}}{nh}$. The probability that the median lies between ξ and $\xi + \frac{\sqrt{\pi}}{nh}$ is therefore

$$\sqrt{\left(\frac{2}{\pi n}\right)} e^{-\frac{2nh^2\xi^2}{\pi}} \cdot \frac{nh}{\sqrt{\pi}} d\xi = \frac{\sqrt{(2n)}h}{\pi} e^{-\frac{2nh^2\xi^2}{\pi}} d\xi.$$

Thus the probability that the median lies at a distance between ξ and $\xi + d\xi$ from the true value is

$$\frac{\mathrm{H}}{\sqrt{\pi}} e^{-\mathrm{H}^2\xi^2} d\xi,$$

where $\mathrm{H} = \frac{\sqrt{(2n)}h}{\sqrt{\pi}}$.

For this result we see that the modulus of precision for the determination of the median is $h\sqrt{\left(\frac{2n}{\pi}\right)}$, and therefore (§ 94) the probable error of the median is

$$\frac{\rho}{h}\sqrt{\left(\frac{\pi}{2n}\right)},$$

where $\rho = 0{\cdot}476936$.

In the last section we have seen that the probable error of the arithmetic mean is $\frac{\rho}{h\sqrt{n}}$. Thus *the error to be feared when we take the median as the true value is* $\sqrt{\left(\frac{\pi}{2}\right)}$ *or* 1·253 *times the error to be feared when we take the true value to be the arithmetic mean.*

103. Accuracy of the Determinations of the Modulus of Precision and Standard Deviation.* — Denote the n deviations by $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and the modulus of precision by h; on the hypothesis that h has a value H, the *a priori* probability of the occurrence of this set of deviations is

$$\frac{\mathrm{H}^n}{(\sqrt{\pi})^n} e^{-\mathrm{H}^2(\epsilon_1^2 + \ldots + \epsilon_n^2)}, \qquad (1)$$

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* Gauss, *Werke*, 4, p. 109; cf. R. A. Fisher, *Month. Not. R.A.S.* 80, p. 758.

while on the hypothesis that h has a value $(\mathrm{H}+\lambda)$ the *a priori* probability of this set of observations is

$$\frac{(\mathrm{H}+\lambda)^n}{(\sqrt{\pi})^n}e^{-(\mathrm{H}+\lambda)^2(\epsilon_1^2+\ldots+\epsilon_n^2)}. \tag{2}$$

By the Principle of Inductive Probability, the ratio of the probability that $(\mathrm{H}+\lambda)$ is the true value of h to the probability that H is the true value of h is equal to the ratio of the expressions (2) and (1); that is, of

$$\left(1+\frac{\lambda}{\mathrm{H}}\right)^n e^{-(2\lambda\mathrm{H}+\lambda^2)(\epsilon_1^2+\ldots+\epsilon_n^2)} \text{ to unity.} \tag{3}$$

Now let H be the most probable value of h (*i.e.* the value which makes (1) a maximum), so that

$$\mathrm{H}=\sqrt{\left\{\frac{n}{2(\epsilon_1^2+\ldots+\epsilon_n^2)}\right\}}.$$

Then (3) may be written

$$\left(1+\frac{\lambda}{\mathrm{H}}\right)^n e^{-(2\lambda\mathrm{H}+\lambda^2)\frac{n}{2\mathrm{H}^2}} \text{ to unity,}$$

or
$$e^{-\frac{\lambda n}{\mathrm{H}}\left(1+\frac{\lambda}{2\mathrm{H}}\right)+n\log\left(1+\frac{\lambda}{\mathrm{H}}\right)} \text{ to unity,}$$

or
$$e^{-\frac{n\lambda^2}{\mathrm{H}^2}+\frac{n\lambda^3}{3\mathrm{H}^3}-\ldots} \text{ to unity,}$$

or, approximately (λ being very small compared with H),

$$e^{-\frac{n\lambda^2}{\mathrm{H}^2}} \text{ to } 1.$$

Therefore the probability that the value of h lies between $\mathrm{H}+\lambda$ and $\mathrm{H}+\lambda+d\lambda$ is nearly

$$\mathrm{K}e^{-\frac{n\lambda^2}{\mathrm{H}^2}}d\lambda,$$

where K is a constant, which, since $\int_{-\infty}^{\infty}\mathrm{K}e^{-\frac{n\lambda^2}{\mathrm{H}^2}}d\lambda=1$, is given by $\mathrm{K}=\frac{1}{\mathrm{H}}\sqrt{\frac{n}{\pi}}$.

Therefore the probability that the value of h lies between $\mathrm{H}+\lambda$ and $\mathrm{H}+\lambda+d\lambda$ is

$$\frac{1}{\mathrm{H}}\sqrt{\frac{n}{\pi}}\cdot e^{-\frac{n\lambda^2}{\mathrm{H}^2}}d\lambda,$$

or the *modulus of precision for the determination of h by the root-mean-square method is* $\sqrt{n}/h$.

From this we deduce at once that *the probability that the standard deviation* σ, *as deduced by the root-mean-square method, lies between* $\sigma + x$ and $\sigma + x + dx$ is

$$\frac{\sqrt{n}}{\sigma\sqrt{\pi}} e^{-\frac{nx^2}{\sigma^2}} dx,$$

so the *modulus of precision for the determination of the standard deviation* σ *by the root-mean-square-method is* $\sqrt{n}/\sigma$. Hence the probable error of the standard deviation σ, as deduced by the root-mean-square method, is

$$\frac{0\cdot476936}{\sqrt{n}}\sigma.$$

Gauss * extended this by showing that the probable error of the standard deviation σ, when it has been deduced by computing the pth powers of the errors, is

$$\frac{0\cdot4769363\sigma\sqrt{2}}{p\sqrt{n}}\left\{\frac{\sqrt{\pi}\,\Gamma(p+\frac{1}{2})}{\Gamma^2\left(\frac{p+1}{2}\right)} - 1\right\}^{\frac{1}{2}}.$$

This gives the following results:

When σ has been computed from the—

I. 1st powers of the errors, the probable error $= \frac{0\cdot5095841}{\sqrt{n}}\sigma$,

II. 2nd „ „ „ „ $= \frac{0\cdot4769363}{\sqrt{n}}\sigma$,

III. 3rd „ „ „ „ $= \frac{0\cdot4971987}{\sqrt{n}}\sigma$,

IV. 4th „ „ „ „ $= \frac{0\cdot5507186}{\sqrt{n}}\sigma$,

V. 5th „ „ „ „ $= \frac{0\cdot6355080}{\sqrt{n}}\sigma$,

VI. 6th „ „ „ „ $= \frac{0\cdot7557764}{\sqrt{n}}\sigma$.

It is evident, therefore, that the most advantageous method is the root-mean-square. In fact, 100 errors of observation yield by the computation of the root-mean-square as good a value of the standard deviation as 114 treated by I., 109 by III., 133 by IV., 178 by V., 251 by VI. There is not much difference as regards accuracy between I. and II., and of course I. is much more convenient for calculation.

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* *Werke*, 4, p. 109.

Lastly, we must consider the accuracy of the determination of the quartile by arranging the n errors of observation according to their absolute magnitude, and taking the middle one as Q; or, more generally, arranging the n errors of observation according to their absolute magnitude, and then taking the error x_p which has m_0 errors less than it, and deriving a value H for the modulus of precision from the equation

$$\frac{m_0}{n} = \Phi(\mathrm{H}x_p).$$

Let h be the true value of the modulus of precision and let x be such that $hx = \mathrm{H}x_p$. Then x is the position that x_p would have if the number of measures were infinite, so that perfect accuracy in the determination of x_p could be attained. Writing $\frac{m_0}{n} = p$, $1 - p = q$, we can show that the probability that out of n measures (all taken positively) $m_0 + r$ lie between 0 and x is

$$\frac{1}{\sqrt{(2\pi npq)}} e^{-\frac{r^2}{2npq}}.$$

Now the probability that a measure (taken positively) lies between x and $x + \xi$, where ξ is small, is

$$\frac{2h}{\sqrt{\pi}} e^{-h^2x^2} \xi,$$

and therefore of n measures (taken positively) the number between x and $x + \xi$ is

$$\frac{2nh}{\sqrt{\pi}} e^{-h^2x^2} \xi.$$

If this number is r, we have

$$r = \frac{2nh}{\sqrt{\pi}} e^{-h^2x^2} \xi,$$

so the probability that $x_p = x + \xi$ is

$$\frac{1}{\sqrt{(2\pi npq)}} e^{-\frac{2nh^2\xi^2}{pq\pi} e^{-2h^2x^2}}$$

and therefore (as in the corresponding discussion of the median, § 102) the modulus of precision for the determination of x_p is $\frac{\sqrt{(2n)}he^{-h^2x^2}}{\sqrt{(pq\pi)}}$. Since $\mathrm{H}x_p = \text{constant}$, we have

$$\mathrm{H}dx_p + x_p d\mathrm{H} = 0,$$

and therefore by the formula for the precision of a linear function of deviations, we have

Modulus of precision for the determination of h by this method

$= \frac{x}{h}$ times the modulus of precision for the determination of x_p

$= \frac{\sqrt{(2n)}xe^{-h^2x^2}}{\sqrt{(pq\pi)}}.$

Hence the square of the standard deviation for the determination of h by this method is

$$\frac{pq\pi}{4nx^2}e^{2h^2x^2};$$

or if t be the value of hx obtained from the equation $\frac{m_0}{n} = \Phi(t)$, it is

$$\frac{h^2}{2n}\,\frac{\pi\Phi(t)\{1-\Phi(t)\}}{2t^2}e^{2t^2}.$$

In particular, when we determine the quartile by finding the deviation which is in the middle of the series of deviations arranged in absolute order of magnitude, we have $\Phi(t) = p = q = \frac{1}{2}$, $t = \rho$ where $\rho = 0{\cdot}476936$, and therefore the standard deviation for the determination of h by this method is

$$\sqrt{\frac{\pi}{n}}\cdot\frac{h}{4\rho}e^{\rho^2},$$

so the probable error of the determination of the quartile by this method is

$$\rho\sqrt{2}\cdot\sqrt{\frac{\pi}{n}}\cdot\frac{\mathrm{Q}}{4\rho}e^{\rho^2}$$

or

$$\sqrt{\frac{\pi}{8n}}\cdot e^{\rho^2}\cdot\mathrm{Q},$$

or, in numbers,

$$\frac{0{\cdot}79}{\sqrt{n}}\mathrm{Q}.$$

This result is due to Gauss:[*] it shows that (on the average of a large number of determinations) 249 measures, treated by this method, must be taken in order to yield as good a value of Q as 100 measures treated by the root-mean-square method: this determination is nearly the same in accuracy as that by the sums of the 6th powers of the errors. We can, however, choose m_0 much more advantageously than this: in fact, determining the minimum of the function

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[*] *Loc. cit.*

$$\frac{\Phi(t)\{1-\Phi(t)\}}{t^2}e^{2t^2},$$

we find that it has a minimum when $t=$ about 1·05, and therefore the most accurate determination of the standard deviation is obtained when we determine the error x, which is such that about 86 per cent of the errors (all taken positively) lie below x and about 14 per cent above x: the probable error of the standard deviation found from this value of x is then only 1·24 times as great as the probable error of the standard deviation determined by the root-mean-square method.* By taking $t=1$ we obtain the following easily remembered precept: *the measure of precision is the reciprocal of that deviation which is exceeded (in absolute value) by 16 per cent of the observed deviations and not attained by 84 per cent of them:* for these percentages we can put more simply $\frac{1}{6}$ and $\frac{5}{6}$.

104. Determination of Probable Error from Residuals.—In § 94 we have regarded a and σ simply as two parameters which occur in the problem of fitting a curve of the type

$$y=\frac{1}{\sigma\sqrt{(2\pi)}}e^{-\frac{(x-a)^2}{2\sigma^2}}$$

to certain data. When, however, we are dealing with errors of measurement of an observed quantity, it is necessary to regard the problem from a somewhat different point of view. We must now take into consideration the fact that the quantity measured has a certain *true value* which, though unknown, must be regarded as possessing a physical existence: this true value must be distinguished from the arithmetic mean of the measures, which is merely the best estimate we can form of it. The differences of the measures from the true value of the quantity are the *errors*, while the differences of the measures from their arithmetic mean are called the *residuals.* We shall now show that the probability that the error should lie between prescribed values a and b is not equal to the probability that a residual should lie between a and b.

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* This was pointed out by F. Hausdorff, *Leipzig Ber.* 53 (1901), p. 164.

Let the measures be denoted by M_1, M_2, . . ., their arithmetic mean by m_0, the true value by m, and let the residuals be

$$v_1 = m_0 - M_1, \quad v_2 = m_0 - M_2, \ . \ . \ . \ ,$$

while the errors are

$$\epsilon_1 = m - M_1, \quad \epsilon_2 = m - M_2, \ . \ . \ . \ .$$

Adding the last equations we have (denoting the sum of n quantities by square brackets)

$$[\epsilon] = nm - [M] = nm - nm_0.$$

Therefore $$m_0 = m - \frac{[\epsilon]}{n}$$

and $$v_1 = m - \frac{[\epsilon]}{n} - M_1 = \epsilon_1 - \frac{[\epsilon]}{n},$$

or $$v_1 = \frac{n-1}{n}\epsilon_1 - \frac{1}{n}\epsilon_2 - \ . \ . \ . \ - \frac{1}{n}\epsilon_n.$$

Thus the residual v_1 is expressed as a linear function of the errors.

Now let h denote the modulus of precision of the errors, and h' the modulus of precision for the residuals: then the formula for the precision of a linear function of errors gives

$$\frac{1}{h'^2} = \left(\frac{n-1}{n}\right)^2 \frac{1}{h^2} + (n-1)\frac{1}{n^2h^2} = \frac{n-1}{nh^2},$$

so the probability that a residual lies between v and $v + dv$ is

$$\sqrt{\left(\frac{n}{n-1}\right)} \cdot \frac{h}{\sqrt{\pi}} e^{-\frac{nh^2v^2}{n-1}} dv.$$

Since $$\frac{1}{2h'^2} = \frac{[v^2]}{n} \text{ and } \frac{1}{2h^2} = \sigma^2,$$

we have therefore

$$\sigma = \sqrt{\left(\frac{[v^2]}{n-1}\right)},$$

and the probable error or quartile of the errors is

$$Q = 0{\cdot}67449\sigma$$
$$= 0{\cdot}67449\sqrt{\left(\frac{[v^2]}{n-1}\right)},$$

and therefore the standard deviation of the arithmetic mean of the n observations, which is $\frac{\sigma}{\sqrt{n}}$, is $\sqrt{\left(\frac{[v^2]}{n(n-1)}\right)}$, while the probable error of the arithmetic mean is $0{\cdot}67449\sqrt{\left(\frac{[v^2]}{n(n-1)}\right)}$. These are generally known as *Bessel's formulae.*

Similarly, the mean absolute deviation η is given in terms of the absolute values of the residuals by the equation

$$\eta = \frac{[|v|]}{\sqrt{\{n(n-1)\}}},$$

so that in terms of the absolute values of the residuals we have

$$\sigma = \sqrt{\frac{\pi}{2}}\eta = \sqrt{\left(\frac{\pi}{2n(n-1)}\right)}[|v|],$$

and the probable error of a single observation is

$$Q = 0{\cdot}84535\eta = \frac{0{\cdot}84535[|v|]}{\sqrt{\{n(n-1)\}}},$$

while the probable error of the arithmetic mean of n observations is $n^{-1}(n-1)^{-\frac{1}{2}}\,0{\cdot}84535[|v|]$.

This formula is due to C. A. F. Peters.* It can be more readily computed than Bessel's and is in general sufficiently accurate.

105. Effect of Errors of Observation on Frequency Curves.—Let a large number N of individuals be measured as regards some attribute, and suppose that the number found to have measures between x and $x+dx$ is $\mathrm{N}y(x)dx$. Suppose, however, that the measures are known to be vitiated to some extent by errors of observation, each measure being liable to error with a modulus of precision h; and suppose that, in consequence of these errors, the number of individuals having a true measure between x and $x+dx$ is not $\mathrm{N}y(x)dx$ but

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* *Ast. Nach.* **44** (1856), p. 29.

N$u(x)dx$. It is required to find the function $u(x)$, the function $y(x)$ and the modulus h being known.

There are actually N$u(t)dt$ individuals having the measure between t and $t+dt$. In consequence of the errors of measurement, these contribute a number

$$\mathrm{N}u(t)dt\frac{h}{\sqrt{\pi}}e^{-h^2(t-x)^2}dx$$

to the measures between x and $x+dx$. Therefore the number of measures observed between x and $x+dx$ is

$$\frac{\mathrm{N}h}{\sqrt{\pi}}dx\int_{-\infty}^{\infty}e^{-h^2(t-x)^2}u(t)dt,$$

or $$\frac{\mathrm{N}h}{\sqrt{\pi}}dx\int_{-\infty}^{\infty}e^{-h^2s^2}u(x+s)ds,$$

or $$\frac{\mathrm{N}h}{\sqrt{\pi}}dx\int_{-\infty}^{\infty}e^{-h^2s^2}\left\{u(x)+su'(x)+\frac{s^2}{2!}u''(x)+\ldots\right\}ds,$$

or $$\mathrm{N}dx\left\{u(x)+\frac{1}{4h^2}u''(x)+\frac{1}{32h^4}u^{\mathrm{iv}}(x)+\ldots\right\}.$$

Thus the functions $y(x)$ and $u(x)$ are connected by the relation

$$y(x)=u(x)+\frac{1}{4h^2}u''(x)+\frac{1}{32h^4}u^{\mathrm{iv}}(x)+\ldots,$$

which readily inverts into

$$u(x)=y(x)-\frac{1}{4h^2}y''(x)+\frac{1}{32h^4}y^{\mathrm{iv}}(x)+\ldots.$$

This equation determines $u(x)$ in terms of $y(x)$.

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

1. The following frequency distribution was obtained by counting the number of letters per line of a book. Calculate the mean value and the standard deviation, and indicate what might be expected to happen as the number of observations is increased.

Number of letters (n)	32	33	34	35	36	37	38	39
Frequency (f)	1	2	2	10	23	31	42	54
(n)	40	41	42	43	44	45	46	47
(f)	46	55	35	28	16	10	2	2

(Edin. Univ. Honours Exam., 1918.)

2. Compute the mean height and standard deviation from the

following data by the method of summation, verifying the result by the root-mean-square method.

Height in inches (h)	54·5	55·5	56·5	57·5	58·5	59·5	60·5	61·5	62·5	63·5	64·5	65·5	66·5	67·5	68·5	69·5	70·5
Frequency (f)	2	4	13	36	69	159	271	326	366	326	229	157	82	32	15	9	4

3. Calculate the probability that in a given interval of time there will be a given number of "calls" at a telephone exchange, and the probability that a subscriber will be kept waiting a given time.

[Cf. A. K. Erlang, *Nyt Tideskrift for Math.* **20** (1909), p. 33.]

4. A vector ϵ is the resultant of a very large number n of elementary vectors, each of given (small) length, whose directions are distributed at random in all directions in the plane. Show that the probability that the resultant vector ϵ should have a length between r and $r+dr$ is

$$\frac{rdr}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}},$$

where σ is independent of r.

[This result is of importance in the theory of the Brownian motion, and also in connection with the scattering of β-rays by matter; the formula is due to Lord Rayleigh.]

ADDITIONAL REFERENCE

H. R. Hulme, " Report on the Statistical Theory of Errors ", *Monthly Notices R.A.S.* **100** (1940), p. 303.

CHAPTER IX

THE METHOD OF LEAST SQUARES

106. Introduction.—In the present chapter we shall be concerned with one particular kind of frequency distribution, namely, the distribution of the measures of an observed quantity, when these measures differ from each other owing to accidental errors of observation.

The deduction of the normal law of frequency given in the preceding chapter is applicable to this particular distribution; but alternative deductions have been given which depend on special assumptions regarding errors of observation, and which are in the highest degree interesting and worthy of study from the point of view of axiomatics. We shall therefore now make a fresh start with the theory.

107. Legendre's Principle.—In the mathematical discussion of the results of observation, it is required to derive from the data the best or most plausible results which they are capable of affording. When the quantities which are observed directly are functions of several unknown quantities which are to be determined, the problem can generally be reduced (as will be seen later) to a formulation such as the following:

It is required to find values for a set of unknown quantities $x, y, z, \ldots$ *in such a way that a set of given equations*

$$\begin{cases} a_1x + b_1y + c_1z + \ldots + f_1t = n_1, \\ a_2x + b_2y + c_2z + \ldots + f_2t = n_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx + b_sy + c_sz + \ldots + f_st = n_s \end{cases}$$

(called the *equations of condition*) *may be satisfied as nearly as* *possible, when the number s of equations is greater than the number of unknowns x, y, z, . . ., t, and the equations are not strictly compatible with each other.*

By saying that the equations are to be satisfied *as nearly as possible* we mean that the quantities

$$\begin{cases} E_1 = a_1x + b_1y + c_1z + . . . + f_1t - n_1, \\ E_2 = a_2x + b_2y + c_2z + . . . + f_2t - n_2, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \\ E_s = a_sx + b_sy + c_sz + . . . + f_st - n_s, \end{cases}$$

which we shall call the *errors*, are to be as small as possible. We shall, for the present, assume that the equations are equally trustworthy, *i.e.* that the quantity, which is more precisely defined later as *weight*, is the same for each equation.

In 1806 Legendre* suggested for the solution of this problem a principle which may be thus stated: *of all possible sets of values of x, y, z, . . ., the most satisfactory is that which renders the sum of the squares of the errors a minimum;* that is,

$$E_1^2 + E_2^2 + . . . + E_s^2$$

is to be a minimum.

For the present we shall simply accept this as a convenient working principle which serves the intended purpose: later in the chapter (§§ 110, 115) we shall examine different attempts which have been made to deduce it from other principles which have been regarded as more evident or better fitted to serve as fundamental axioms.

108. **Deduction of the Normal Equations.**—Assuming Legendre's principle, we have now to find the values of $x, y, z, . . .$ which make

$$E_1^2 + E_2^2 + . . . + E_s^2$$

a minimum. If we use the notation [ ] as a symbol of summation so that, *e.g.*, $[aa] = a_1^2 + a_2^2 + . . . + a_s^2$, $[ab] = a_1b_1 + a_2b_2 + . . . + a_sb_s$, the sum of squares is

$$[aa]x^2 + [bb]y^2 + [cc]z^2 + . . . + 2[ab]xy + 2[ac]xz + . . . - 2[an]x - 2[bn]y - . . . + [nn].$$

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* *Nouvelles Méthodes pour la détermination des orbites des comètes*, Paris, 1806, p. 72.

The unknowns $x, y, z, \ldots$ are to be chosen to make this a minimum, so that the derivatives with respect to $x, y, z, \ldots$ must vanish. Thus a minimum value is obtained when the unknowns are calculated from the equations

$$\begin{cases} [aa]x + [ab]y + [ac]z + \ldots = [an], \\ [ab]x + [bb]y + [bc]z + \ldots = [bn], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

These are known as the *normal equations*: from them the values of $x, y, z, \ldots$ are to be determined by ordinary algebraic analysis. Evidently, *in order to form the normal equation with respect to any one of the unknowns, we must multiply each of the original equations by the coefficients of the unknown in this equation and add together all these products.*

The method which has been followed was named by Legendre the *Method of Least Squares.*

In deriving the normal equations from the equations of condition we can use tables of quarter squares, for we have not only

$$[aa] = a_1^2 + a_2^2 + \ldots + a_s^2,$$

but also $$[ab] = \tfrac{1}{2}\{[(a+b)(a+b)] - [aa] - [bb]\}.$$

As a check on the computation of the normal equations, it is well to calculate the sums

$$\begin{gathered} \sigma_1 = a_1 + b_1 + c_1 + \ldots + f_1 + n_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sigma_s = a_s + b_s + c_s + \ldots + f_s + n_s, \end{gathered}$$

and from these to calculate the sums $[a\sigma]$, $[b\sigma]$, $\ldots$, $[f\sigma]$. These, with the sums $[aa] \ldots [ff]$ which have already been computed, should satisfy the equations

$$\begin{cases} [aa] + [ab] + \ldots + [af] + [an] = [a\sigma], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [af] + [bf] + \ldots + [ff] + [nf] = [f\sigma]. \end{cases}$$

The verification of these equations serves to assure us of the accuracy of our calculation of the normal equations. We shall discuss later (§§ 117-121) the systematic solution of the normal equations and the controls connected therewith. For the present we shall regard the solution merely as a matter of elementary algebra.

Ex. 1.—*Find the most plausible values of x and y from the equations*

$$\begin{aligned} 4{\cdot}91x - 59{\cdot}0y &= -339{\cdot}8 \\ 2{\cdot}72x - 2{\cdot}7y &= -47{\cdot}5 \\ 0{\cdot}05x + 32{\cdot}4y &= 262{\cdot}5 \\ -2{\cdot}91x + 27{\cdot}7y &= 152{\cdot}9 \\ -4{\cdot}77x + 1{\cdot}4y &= -27{\cdot}9. \end{aligned}$$

We shall first find the normal equation for x by use of Crelle's multiplication table. Multiplying each equation by the coefficient of x in it, we have

$$\begin{aligned} 24{\cdot}11x - 289{\cdot}7y &= -1668{\cdot}4 \\ 7{\cdot}40x - 7{\cdot}3y &= -129{\cdot}2 \\ 1{\cdot}6y &= 13{\cdot}1 \\ 8{\cdot}47x - 80{\cdot}6y &= -444{\cdot}9 \\ 22{\cdot}75x - 6{\cdot}7y &= 133{\cdot}1. \end{aligned}$$

Adding, we get

$$62{\cdot}73x - 382{\cdot}7y = -2096{\cdot}3. \tag{1}$$

This is the normal equation for x.

We shall now find the normal equation for y by use of a table of squares. We have

$$\begin{aligned} [aa] &= 4{\cdot}91^2 + 2{\cdot}72^2 + 2{\cdot}91^2 + 4{\cdot}77^2 = 62{\cdot}73 \\ [bb] &= 59^2 + 2{\cdot}7^2 + 32{\cdot}4^2 + 27{\cdot}7^2 + 1{\cdot}4^2 = 5307{\cdot}3 \\ [a+b,\ a+b] &= 54{\cdot}09^2 + 32{\cdot}45^2 + 24{\cdot}79^2 + 3{\cdot}37^2 \\ &= 2925{\cdot}73 + 1053 + 614{\cdot}54 + 11{\cdot}36 \\ &= 4604{\cdot}63 \\ [b+n,\ b+n] &= 281846{\cdot}10 \\ [n,\ n] &= 210783{\cdot}36. \end{aligned}$$

Therefore

$$\begin{aligned} [ab] &= \tfrac{1}{2}\{4604{\cdot}63 - 62{\cdot}73 - 5307{\cdot}3\} = -382{\cdot}7 \\ [bn] &= \tfrac{1}{2}\{[b+n,\ b+n] - [bb] - [nn]\} = \tfrac{1}{2}(65755{\cdot}44) = 32877{\cdot}7, \end{aligned}$$

and the normal equation of y is

$$-382{\cdot}7x + 5307{\cdot}3y = 32877{\cdot}7. \tag{2}$$

From (1) and (2) we find

$$x = 7{\cdot}81,\ y = 6{\cdot}76.$$

The computation of the normal equations and the check on the computation may be carried out simultaneously as in the following scheme: the coefficients in the normal equation for x being read from the row (3), and the coefficients in the normal equation for y from the row (4). In each table a row is formed for each of the given equations, and the columns are added. The sum of the first three columns should then be equal to the sum of the last column.

a.	*b*.	*n*.	σ
4·91	– 59·0	– 339·8	– 393·89
2·72	– 2·7	– 47·5	– 47·48
0·05	32·4	262·5	294·95
– 2·91	27·7	152·9	177·69
– 4·77	1·4	– 27·9	– 31·27

	aa.	*ab*.	*an*.	$a\sigma$.	
	24·1081	– 289·690	– 1668·418	– 1933·9999	
	7·3984	– 7·344	– 129·200	– 129·1456	
	0·0025	1·620	13·125	14·7475	
	8·4681	– 80·607	– 444·939	– 517·0779	
	22·7529	– 6·678	133·083	149·1579	
Sum	62·7300	– 382·699	– 2096·349	– 2416·3180	(3)

	ba.	*bb*.	*bn*.	$b\sigma$.	
	– 289·69	3481·00	20048·20	23239·51	
	– 7·344	7·29	128·25	128·196	
	1·62	1049·76	8505·00	9556·38	
	– 80·607	767·29	4235·33	4922·013	
	– 6·678	1·96	– 39·06	– 43·778	
Sum	– 382·699	5307·30	32877·72	37802·321	(4)

	na.	*nb*.	*nn*.	$n\sigma$.	
	– 1668·418	20048·20	115464·04	133843·822	
	– 129·200	128·25	2256·25	2255·300	
	13·125	8505·00	68906·25	77424·375	
	– 444·939	4235·33	23378·41	27168·801	
	133·083	– 39·06	778·41	872·433	
Sum	– 2096·349	32877·72	210783·36	241564·731	(5)

Ex. 2.—*Find the most plausible values of x and y from the equations*

$$\begin{aligned} x+\ \ y &= 3{\cdot}01\\ 2x-\ \ y &= 0{\cdot}03\\ x+3y &= 7{\cdot}02\\ 3x+\ \ y &= 4{\cdot}97. \end{aligned}$$

Ex. 3.—*The following data were read from a graph showing the probable stature of son for given father's stature, denoted by S inches and F inches respectively:*

S.	65·7	66·8	67·2	69·3	69·8	70·5	70·9
F.	62	64	65	69	70	71	72.

If S and F are connected by the relation $S=a+bF$, *determine by the Method of Least Squares the most probable values of the constants a and b.*
(Edin. Univ. Actuarial Diploma, 1922.)

109. **Reduction of the Equations of Condition to the Linear Form.**—In many cases the original data of the problem are not immediately expressible as a set of linear equations of condition. Suppose, for example, we require to find the most plausible values of x and y from a set of equations

$$f_1(x, y)=u_1,\quad f_2(x, y)=u_2,\quad \ldots,\ f_s(x, y)=u_s,$$

where $f_1, \ldots, f_s$ are known functions and $u_1, u_2, \ldots, u_s$ are measures derived from observation and liable to accidental error. In this case we first find, in any way, an approximate pair of values of x and y, say $\bar{x}, \bar{y}$. Putting $x=\bar{x}+\xi$, $y=\bar{y}+\eta$, we have

$$f_1(x, y)=f_1(\bar{x}+\xi, \bar{y}+\eta)=f_1(\bar{x}, \bar{y})+\frac{\partial f_1(\bar{x}, \bar{y})}{\partial \bar{x}}\xi+\frac{\partial f_1(\bar{x}, \bar{y})}{\partial \bar{y}}\eta$$

approximately.

The equations of condition thus become

$$\begin{cases} \dfrac{\partial f_1(\bar{x}, \bar{y})}{\partial \bar{x}}\xi+\dfrac{\partial f_1(\bar{x}, \bar{y})}{\partial \bar{y}}\eta=u_1-f_1(\bar{x}, \bar{y}),\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \dfrac{\partial f_s(\bar{x}, \bar{y})}{\partial \bar{x}}\xi+\dfrac{\partial f_s(\bar{x}, \bar{y})}{\partial \bar{y}}\eta=u_s-f_s(\bar{x}, \bar{y}), \end{cases}$$

and these, being linear in (ξ, η), can be solved by the process already described.

Ex.—*Determine the most plausible values of the rectangular coordinates* (x, y) *of a point P, given that its measured distances from the points* (0, 0) (7, 0) (0, 6) *are respectively* 6·40, 4·47, 5·38, *and that these measures are of equal weight.*

We readily find graphically that approximate values are $\bar{x}=5$, $\bar{y}=4$. Therefore writing $x=\bar{x}+\xi$, $y=\bar{y}+\eta$, the equations of condition become

$$\begin{aligned}\{(5+\xi)^2+(4+\eta)^2\}^{\frac{1}{2}}&=6{\cdot}40,\\ \{(2-\xi)^2+(4+\eta)^2\}^{\frac{1}{2}}&=4{\cdot}47,\\ \{(5+\xi)^2+(2-\eta)^2\}^{\frac{1}{2}}&=5{\cdot}38,\end{aligned}$$

or

$$\left\{\begin{aligned}\frac{5\xi+4\eta}{\sqrt{41}}&=6{\cdot}40-\sqrt{41}=-0{\cdot}0031,\\ \frac{-2\xi+4\eta}{\sqrt{20}}&=4{\cdot}47-\sqrt{20}=-0{\cdot}0021,\\ \frac{5\xi-2\eta}{\sqrt{29}}&=5{\cdot}38-\sqrt{29}=-0{\cdot}0052.\end{aligned}\right.$$

These equations, which are linear in ξ and η, are now treated as ordinary equations of condition as in § 108.

110. Gauss's "Theoria Motus"; the Postulate of the Arithmetic Mean.

—We now proceed to consider the various attempts that have been made to place the Method of Least Squares on a logical foundation.

The first writer to connect the method with the mathematical theory of probability was Gauss.[*] His treatment assumes as a postulate that *when any number of equally good direct observations* M, M′, M″, . . . *of an unknown magnitude x are given, the most probable value is their arithmetic mean.* Gauss's deduction of the law of error will be given in § 112: for the present we shall consider the postulate in itself.[†]

This postulate must be distinguished from the statement that as the number of observations is increased indefinitely, the arithmetic mean tends to the true value of x: this latter statement is indeed correct,[‡] and is true of an infinite number of other functions besides the arithmetic mean:[§] but we cannot infer from it that the arithmetic mean gives the *most probable* result when the number of observations is finite.

In recent years the postulate of the arithmetic mean has been exhibited as a deduction from other axioms of a more elementary nature,* which may be formulated thus:

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[*] *Theoria motus corporum cœlestium,* Hamburg (1809), § 177. Gauss mentions that he had used the method from the year 1795.

[†] For a critical discussion see P. Pizzetti, "I fondamenti mat. per la critica dei risultati sperimentali," *Atti della R. Univ. di Genova per il centenario Colombiano,* 1892, pp. 113-334.

[‡] Indeed we may *define* the *true value* of a physical quantity as *the limit to which the mean of n observations tends when n increases indefinitely.*

[§] *E.g.* $\frac{\Sigma f(\mathrm{M}-x)}{n}\to 0$, where f is any *odd* function, when the number of observations is increased indefinitely.

Axiom I.—The differences between the most probable value and the individual measures do not depend on the position of the null-point from which they are reckoned.

Axiom II.—The ratio of the most probable value to any individual measure does not depend on the unit in terms of which measures are reckoned.

Axiom III.—The most probable value is independent of the order in which the measurements are made, and so is a symmetric function of the measures.

Axiom IV.—The most probable value, regarded as a function of the individual measures, has one-valued and continuous first derivatives with respect to them.

From these four axioms we can derive the postulate of the arithmetic mean in the following way:

Suppose the most probable value is expressed in terms of the n measures $x_1, x_2, \ldots, x_n$ by the function $f(x_1, x_2, \ldots, x_n)$. Then by the theorem of the mean value in the differential calculus (which by Axiom IV. is applicable), we have

$$f(kx_1, kx_2, \ldots, kx_n) = f(0, 0, \ldots, 0) + kx_1\left[\frac{\partial f}{\partial x_1}\right] + \ldots + kx_n\left[\frac{\partial f}{\partial x_n}\right],$$

where the square brackets denote that every x is to be replaced by θkx, where θ lies between 0 and 1. Now by Axiom II., the left-hand side $= kf(x_1, x_2, \ldots, x_n)$: and since by the continuity of the function f, the equation

$$f(kx_1, kx_2, \ldots, kx_n) = kf(x_1, x_2, \ldots, x_n)$$

must hold in the limit when k is zero, we have

$$f(0, 0, \ldots, 0) = 0.$$

Thus we have

$$kf(x_1, x_2, \ldots, x_n) = kx_1\left[\frac{\partial f}{\partial x_1}\right] + \ldots + kx_n\left[\frac{\partial f}{\partial x_n}\right],$$

or, dividing by k,

$$f(x_1, x_2, \ldots, x_n) = x_1\left[\frac{\partial f}{\partial x_1}\right] + \ldots + x_n\left[\frac{\partial f}{\partial x_n}\right].$$

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* Cf. G. Schiaparelli, *Rend. Ist. Lombardo*, (2) **40** (1907), p. 752, and *Ast. Nach.* **176** (1907), p 205; U. Broggi, *L'Enseignement mathématique*, **11** (1909), p. 14; R. Schimmack, *Math. Ann.* **68** (1909), p. 125.

In this equation make $k \to 0$: then each of the quantities $\left[\frac{\partial f}{\partial x_n}\right]$ tends to a value which is independent of the x's, so we can write $f(x_1, x_2, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$, where the c's are independent of the x's. By Axiom III., the c's must all be equal, so

$$f(x_1, x_2, \ldots, x_n) = c(x_1 + x_2 + \ldots + x_n),$$

and since Axiom I. gives the equation

$$f(x_1 + h, \ldots, x_n + h) = f(x_1, \ldots, x_n) + h,$$

we have

$$cnh = h,$$

so

$$c = \frac{1}{n}.$$

Therefore $$f(x_1, x_2, \ldots, x_n) = \frac{1}{n}(x_1 + x_2 + \ldots + x_n),$$

which expresses the postulate of the arithmetic mean.

111. **Failure of the Postulate of the Arithmetic Mean.**—For certain types of observations the postulate of the arithmetic mean is not valid: in particular, for visual photometric measurements in astronomy.* In these the quantity, of which measures are made, is the ratio of the brightnesses of the two stars. Suppose that x is the true value of the ratio for the two particular stars, and let $l_1, l_2, \ldots, l_n$ be different measures of it. The observations being supposed to be made visually, we take as our starting-point the Weber-Fechner psycho-physical law on the sensitiveness of the human retina to differences of light intensity: this asserts that increment of sensation is proportional to relative increase of excitation, or $\delta E = \text{constant} \times \frac{\delta l}{l}$, where E measures the sensation of light and l is a physical measure of its intensity. If E and E_0 denote the intensities of perception corresponding to the brightnesses l and x, we have therefore

$$E - E_0 = c \log \frac{l}{x}.$$

The quantities $E - E_0$ represent the errors of observation; denoting them by $\Delta_1, \Delta_2, \ldots, \Delta_n$, we have

$$\Delta_1 = c \log \frac{l_1}{x}, \quad \Delta_2 = c \log \frac{l_2}{x}, \quad \ldots \quad \Delta_n = c \log \frac{l_n}{x}.$$

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* Cf. Seeliger, *Ast. Nach.* 132 (1893), col. 209.

The Δ's appear to obey the normal law of facility: so that the most probable value of x is that which makes

$$e^{-h^2(\Delta_1^2+\Delta_2^2+\ldots+\Delta_n^2)}$$

a maximum, *i.e.* which makes

$$\Delta_1^2+\Delta_2^2+\ldots+\Delta_n^2$$

a minimum; this gives

$$\Delta_1\frac{\partial\Delta_1}{\partial x}+\Delta_2\frac{\partial\Delta_2}{\partial x}+\ldots+\Delta_n\frac{\partial\Delta_n}{\partial x}=0$$

or
$$\log\frac{l_1}{x}+\log\frac{l_2}{x}+\ldots+\log\frac{l_n}{x}=0,$$

so
$$\frac{l_1l_2\ldots l_n}{x^n}=1,$$

or
$$x=(l_1l_2\ldots l_n)^{\frac{1}{n}}.$$

This formula for determining the most probable value from the observations was first given by Seidel in 1863.*

Ex.—Show that if the probability that the error of a measurement will be between x and $x+dx$ is $c\{1+|hx|^{\frac{3}{2}}\}^{-1}dx$, where c and h are constant, then the arithmetic mean of two measurements is not as reliable as a single measurement, and in fact is the least probable value among all possible weighted means. (E. L. Dodd.)

112. Gauss's "Theoria Motus" Proof of the Normal Law.

—We shall now show how, when the postulate of the arithmetic mean is granted, the normal law of error can be deduced.

Suppose that for the measure of an observed quantity, the probability of an error between Δ and $\Delta+d\Delta$ is $\phi(\Delta)d\Delta$, so that $\phi(\Delta)$ is the relative frequency of error. If ϵ denotes the least quantity to which the measuring-instrument is sensitive, we can suppose that the possible values of any measure proceed by steps of amount ϵ, and the probability of an error Δ may be taken as $\phi(\Delta)\epsilon$.

It should be noticed that it is here tacitly assumed that the probability of a certain deviation depends only on the magnitude of the deviation. If this assumption is not made, a law of facility much more general than the normal law may be deduced.†

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* *München. Abh.* 9 (1863).

† Cf. Poincaré, *Calcul. des prob.* p. 155, and B. Meidell, *Zeits. für Math. u. Phys.* 56 (1908), p. 77.

Now suppose that a number s of measures M, M', M'', . . . are taken of a quantity x whose true value is p. The errors are $\Delta = M - p$, $\Delta' = M' - p$, etc. The probability of the error in the first measurement being $M - p$ is $\phi(M - p)\epsilon$; the probability of the error in the second measurement being $M' - p$ is $\phi(M' - p)\epsilon$, and so on. The probability that a set of measures $(M, M', M'' \ldots)$ will occur is therefore

$$\epsilon^s . \phi(M - p)\phi(M' - p)\phi(M'' - p) \ldots .$$

If now we assume that, before the observations are made, all values of x are equally likely to be the true value,* it follows from Bayes' theorem in Inductive Probability that, when the observations have been made, the probability of the true value of x lying between p and $p + dp$ is

$$\frac{\phi(M - p)\phi(M' - p)\phi(M'' - p) \ldots dp}{\int_{-\infty}^{\infty} \phi(M - p)\phi(M' - p)\phi(M'' - p) \ldots dp},$$

and therefore the *most probable* hypothesis regarding the true value of x is that x has that value which makes

$$\phi(M - x)\phi(M' - x)\phi(M'' - x) \ldots$$

a maximum, *i.e.* that value of x for which

$$\Sigma \frac{d}{dx} \log \phi(M - x) = 0. \qquad (1)$$

Adopting the postulate of the arithmetic mean, we see that equation (1) must be equivalent to

$$x = \frac{1}{n}(M + M' + \ldots)$$

or $$\Sigma(M - x) = 0.$$

Therefore $$\frac{d}{dx} \log \phi(M - x) = c(M - x),$$

where c denotes a constant, and therefore

$$\phi(M - x) = Ae^{-\frac{1}{2}c(M-x)^2},$$

where A denotes a constant, so

$$\phi(\Delta) = Ae^{-\frac{1}{2}c\Delta^2}.$$

---

* It is not necessary to make this an independent assumption, for it may be deduced as a consequence of the postulate of the arithmetic mean, which will be introduced presently.

Since the sum of the probabilities of all possible errors is unity, we have

$$\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = 1$$

or

$$1 = \mathrm{A}\int_{-\infty}^{\infty} e^{-\frac{1}{2}c\Delta^2}d\Delta.$$

But

$$\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi},$$

so

$$\mathrm{A} = \sqrt{\frac{c}{2\pi}}.$$

Writing h for $\sqrt{(\frac{1}{2}c)}$, we have

$$\phi(\Delta) = \frac{h}{\sqrt{\pi}}e^{-h^2\Delta^2},$$

which shows that *the distribution of the measures about the true value is a normal frequency distribution.*

113. **Gauss's "Theoria Motus" Discussion of Direct Measurements of a Single Quantity.**—Assuming, then, that in the measure of an observed quantity the probability of an error between Δ and $\Delta + d\Delta$ is

$$\frac{h}{\sqrt{\pi}}e^{-h^2\Delta^2}d\Delta,$$

where h is the modulus of precision, we note that the modulus of precision affords an indication of the *weight* which must be attached to an observation when it is to be combined with other observations. Thus in observations to determine the time, made with the meridian circle, the modulus of precision is less for a star very near the pole than for an equatorial star, so in combining the results of an equatorial with a circumpolar observation we should attach more importance to one than to the other.

Suppose now that s measurements are made of a quantity x, the measures being x_1, x_2, . . ., x_s, and the corresponding measures of precision being h_1, h_2, . . ., h_s. Let p be the true value of x, so the errors are $p - x_1$, $p - x_2$, . . . The probability of precisely this set of errors is therefore

$$\frac{h_1\epsilon_1}{\sqrt{\pi}}e^{-h_1^2(p-x_1)^2} \cdot \frac{h_2\epsilon_2}{\sqrt{\pi}}e^{-h_2^2(p-x_2)^2} \cdot \ldots \frac{h_s\epsilon_s}{\sqrt{\pi}}e^{-h_s^2(p-x_s)^2}.$$

Now the most probable value of x is that value of p which makes this expression a maximum, *i.e.* it is the value of x which makes

$$h_1^2(x-x_1)^2+h_2^2(x-x_2)^2+\ldots+h_s^2(x-x_s)^2$$

a minimum. It is therefore given by

$$h_1^2(x-x_1)+h_2^2(x-x_2)+\ldots+h_s^2(x-x_s)=0,$$

or

$$x=\frac{h_1^2x_1+h_2^2x_2+\ldots+h_s^2x_s}{h_1^2+h_2^2+\ldots+h_s^2}. \tag{1}$$

Now we have seen (§ 89) that the modulus of precision H for any linear function $a_1\Delta_1+a_2\Delta_2+\ldots$ of the errors Δ_1, Δ_2, . . . whose moduli of precision are h_1, h_2, . . . is given by the equation

$$\frac{1}{H^2}=\frac{a_1^2}{h_1^2}+\frac{a_2^2}{h_2^2}+\ldots+\frac{a_s^2}{h_s^2}. \tag{2}$$

Therefore the modulus of precision of the quantity x given by (1) is H when

$$\frac{1}{H^2}=\frac{1}{(\Sigma h^2)^2}\left\{\frac{h_1^4}{h_1^2}+\frac{h_2^4}{h_2^2}+\ldots+\frac{h_s^4}{h_s^2}\right\},$$

so

$$H^2=h_1^2+h_2^2+\ldots+h_s^2. \tag{3}$$

Now let h be the modulus of precision of certain observations which are taken as a standard for comparison of precision, and write

$$w_1=\frac{h_1^2}{h^2},\quad w_2=\frac{h_2^2}{h^2},\ \ldots,\quad w_s=\frac{h_s^2}{h^2},\quad W=\frac{H^2}{h^2}. \tag{4}$$

The equations (1) and (3) become

$$\begin{cases} x=\dfrac{w_1x_1+w_2x_2+\ldots+w_sx_s}{w_1+w_2+\ldots+w_s}, \\ W=w_1+w_2+\ldots+w_s. \end{cases}$$

These equations are evidently analogous to the equations which determine the position of the centre of gravity of particles of weights w_1, w_2, . . ., w_s placed at the points x_1, x_2, . . ., x_s respectively, W denoting the weight of the equivalent body to be placed at the centre of gravity. On this account w_1, w_2, . . ., w_s are called the *weights* of the observations x_1, x_2, . . ., x_s respectively; the weight W of the result is the sum of the weights of the separate observations.

Hence we deduce at once the following results:

1. *The mean of s equally good observations has a weight s times that of any one of them.*

2. *If w denotes the weight of a determination p which is deduced as the most probable value of x from a certain set of observations, and if we adjoin a new observation $x=p+u$ of weight* 1, *then the most probable value of x is $p+\frac{u}{w+1}$, and it has the weight $(w+1)$.*

3. *If an observation x has the weight w, then on multiplying it by any number λ the new value λx has the weight $\frac{w}{\lambda^2}$.* For if the probability that x lies between x and $x+dx$ is $\frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx$, then, denoting λx by y, the probability that y lies between y and $y+dy$ is $\frac{h}{\lambda\sqrt{\pi}}e^{-\frac{h^2}{\lambda^2}y^2}dy$, so the modulus of precision for λx is $\frac{h}{\lambda}$, and therefore the weight of λx is $\frac{w}{\lambda^2}$. It follows that:

4. *Observations of different weights can be treated as observations of unit weight by multiplying each equation of condition by the square root of its weight* (§ 114).

Ex.—If $a_1x_1+a_2x_2+\ldots+a_sx_s$ is a linear function of independent estimates $x_1, x_2, \ldots, x_s$ of a number x, determine $a_1, a_2, \ldots, a_s$ subject to the condition $a_1+a_2+\ldots+a_s=1$, so that the weight of the linear function may be a maximum.

From equations (2) and (4) above we see that $\frac{1}{W}=\frac{a_1^2}{w_1}+\frac{a_2^2}{w_2}+\ldots+\frac{a_s^2}{w_s}$, where W is the weight of the linear function and $w_1, w_2, \ldots, w_s$ are the weights of the independent estimates. Therefore we must have

$$0=\frac{a_1da_1}{w_1}+\frac{a_2da_2}{w_2}+\ldots+\frac{a_sda_s}{w_s}$$

subject to

$$0=da_1+da_2+\ldots+da_s,$$

and therefore

$$\frac{a_1}{w_1}=\frac{a_2}{w_2}=\ldots=\frac{a_s}{w_s}=\frac{a_1+a_2+\ldots+a_s}{w_1+w_2+\ldots+w_s},$$

so

$$a_1=\frac{w_1}{w_1+w_2+\ldots+w_s}.$$

Thus the estimate for x which has the greatest weight is

$$x=\frac{w_1x_1+w_2x_2+\ldots+w_sx_s}{w_1+w_2+\ldots+w_s}.$$

This agrees with the value of x given by the Method of Least Squares.

114. Gauss's "Theoria Motus" Discussion of Indirect Observations.—Now suppose that a set of unknown quantities $x, y, z, \ldots$ are to be determined from s measures $n_1, n_2, \ldots, n_s$, the unknowns being connected with the measured quantities by the equations of condition

$$\begin{cases} a_1x + b_1y + c_1z + \ldots + f_1t = n_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx + b_sy + c_sz + \ldots + f_st = n_s, \end{cases}$$

where the coefficients $a_1, b_1, \ldots, a_s, b_s, \ldots$ are supposed to be known accurately, while the measures are liable to accidental errors of observation. Suppose that for the observed quantity n_1 the measure of precision is h_1, for the observed quantity n_2 the measure of precision is h_2, etc.; then the probability of an error Δ_1 in n_1 is $\frac{h_1\epsilon_1}{\sqrt{\pi}}e^{-h_1^2\Delta_1^2}$, where ϵ_1 is the smallest quantity to which the measuring-instrument of n_1 is sensitive, so that the measures may be supposed to proceed by steps of amount ϵ_1. Then as in § 113, the probability of the concurrence of errors $\Delta_1, \Delta_2, \ldots$ is

$$\frac{h_1\epsilon_1}{\sqrt{\pi}}e^{-h_1^2\Delta_1^2}\frac{h_2\epsilon_2}{\sqrt{\pi}}e^{-h_2^2\Delta_2^2}\ldots,$$

and the most probable values of $x, y, z, \ldots, t$ are those which make this expression a maximum, when

$$\Delta_1 = a_1x + b_1y + c_1z + \ldots + f_1t - n_1,$$
$$\Delta_2 = a_2x + b_2y + c_2z + \ldots + f_2t - n_2, \text{ etc.}$$

The values $(x, y, z, \ldots, t)$ must therefore be determined from the condition that

$$h_1^2\Delta_1^2 + h_2^2\Delta_2^2 + \ldots + h_s^2\Delta_s^2$$

is to be a minimum. If $w_1, w_2, \ldots, w_s$ are the *weights* of the observations (which are proportional to the squares of the moduli of precision), we must therefore have

$$w_1\Delta_1^2 + w_2\Delta_2^2 + \ldots + w_s\Delta_s^2$$

a minimum: that is, in the notation of § 108,

$$[waa]x^2 + [wbb]y^2 + [wcc]z^2 + \ldots + 2[wab]xy + \ldots$$
$$- 2[wan]x - \ldots + [wnn]$$

must be a minimum, and therefore the unknowns $x, y, z,$. must be determined from the equations

$$\begin{cases} [waa]x + [wab]y + [wac]z + \ldots = [wan], \\ [wab]x + [wbb]y + [wbc]z + \ldots = [wbn], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

These are the *normal equations.*

It is evident that an observation of weight w enters into equations exactly as if it were w separate observations each of weight unity. The best practical method of accounting for weight is, however, to prepare the equations of condition by multiplying each equation throughout by the square root of its weight: the resulting equations then have their weights all equal, as we have seen (§ 113).

115. **Laplace's Proof and Gauss's "Theoria Combinationis" Proof.**—The Method of Least Squares was established in an entirely different manner by Laplace in 1811,* and by Gauss (who to a considerable extent adopted Laplace's ideas) in 1821–23.†

The common principle of these and various modern proofs which have been developed from them may be stated thus:

Suppose that for s linear expressions

$$\left.\begin{matrix} a_1x + b_1y + c_1z + \ldots \\ a_2x + b_2y + c_2z + \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx + b_sy + c_sz + \ldots \end{matrix}\right\} \qquad (1)$$

we have respectively the independent estimates $n_1, n_2, \ldots, n_s$, derived from observation, the number s being supposed greater than the number of unknowns $x, y, z, \ldots$.

For the quantity

$$\lambda_1(a_1x + b_1y + c_1z + \ldots) + \lambda_2(a_2x + b_2y + c_2z + \ldots) + \ldots + \lambda_s(a_sx + b_sy + c_sz + \ldots) \quad \ldots \quad (A)$$

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* *Théorie anal. des prob.* Livre II. chap. iv. (1812), following a memoir of 1811.

† *Theoria combinationis observationum erroribus minimis obnoxiae*, *Werke*, Band IV. p. 1. A French translation by J. Bertrand was published at Paris in 1855.

Gauss in a letter to Bessel of February 28, 1839, admitted that he had changed his views regarding the establishment of the Method of Least Squares since the publication of his *Theoria Motus* in 1809, having abandoned the "metaphysical" basis on which the Method was founded in that work.

we have therefore the estimate

$$\lambda_1 n_1 + \lambda_2 n_2 + \ldots + \lambda_s n_s.$$

Suppose the λ's are chosen so that in the expression (A) the resultant coefficient of x is unity and the resultant coefficients of $y, z, \ldots$ are all zero, so that

$$\lambda_1 n_1 + \ldots + \lambda_s n_s$$

is an estimate of x. The problem is to lay down such further conditions for the λ's as will secure that this is the *best possible* estimate for x. Now this can be done if we take as a fundamental idea the notion of the *weight* of an estimate. Let n be an estimate, derived from observation, of the value of some quantity: then we shall suppose that with this estimate is associated a number w which will be called its *weight*. We shall further establish (in various ways according as the method of Laplace or of Gauss or of more modern writers is followed) that if $w_1, w_2, \ldots, w_s$ are the weights of the estimates $n_1, n_2, \ldots, n_s$ for the linear expressions (1), then the weight of the estimate $\lambda_1 n_1 + \ldots + \lambda_s n_s$ for the expression (A) is W, where

$$\frac{1}{W} = \frac{\lambda_1^2}{w_1} + \frac{\lambda_2^2}{w_2} + \ldots + \frac{\lambda_s^2}{w_s}.$$

Finally, we shall define the *best possible* estimate for x to be that whose weight is greatest. With these presuppositions the best estimate for x is

$$x = \lambda_1 n_1 + \ldots + \lambda_s n_s, \tag{B}$$

where the λ's satisfy the equations

$$\left.\begin{array}{l} \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_s a_s = 1 \\ \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_s b_s = 0 \\ \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_s c_s = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}. \tag{C}$$

Also we have

$$\frac{\lambda_1 d\lambda_1}{w_1} + \frac{\lambda_2 d\lambda_2}{w_2} + \ldots + \frac{\lambda_s d\lambda_s}{w_s} = 0.$$

These equations give at once

$$\left\{\begin{array}{l} \dfrac{\lambda_1}{w_1} + \mu a_1 + \mu' b_1 + \mu'' c_1 + \ldots = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\lambda_s}{w_s} + \mu a_s + \mu' b_s + \mu'' c_s + \ldots = 0, \end{array}\right.$$

or

$$\left.\begin{array}{l}\lambda_1 + \mu w_1 a_1 + \mu' w_1 b_1 + \mu'' w_1 c_1 + \ldots = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_s + \mu w_s a_s + \mu' w_s b_s + \mu'' w_s c_s + \ldots = 0\end{array}\right\}. \qquad \text{(D)}$$

Substituting for the λ's from (D) in (B) and (C), we have

$$\begin{aligned} 0 &= x + \mu[wan] + \mu'[wbn] + \mu''[wcn] + \ldots \\ 0 &= 1 + \mu[waa] + \mu'[wba] + \mu''[wca] + \ldots \\ 0 &= \quad \mu[wab] + \mu'[wbb] + \mu''[wcb] + \ldots \\ 0 &= \quad \mu[wac] + \mu'[wbc] + \mu''[wcc] + \ldots \\ &\text{etc.} \end{aligned}$$

Eliminating the μ's from these equations, we have

$$0 = \begin{vmatrix} x & [wan] & [wbn] & \ldots & [wfn] \\ 1 & [waa] & [wba] & \ldots & [wfa] \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & [waf] & [wbf] & \ldots & [wff] \end{vmatrix}$$

or

$$x = \frac{\begin{vmatrix} [wan] & [wbn] & \ldots & [wfn] \\ [wab] & [wbb] & \ldots & [wfb] \\ [wac] & [wbc] & \ldots & [wfc] \\ \cdot & \cdot & \cdot & \cdot \\ [waf] & [wbf] & \ldots & [wff] \end{vmatrix}}{\begin{vmatrix} [waa] & [wba] & \ldots & [wfa] \\ [wab] & [wbb] & \ldots & [wfb] \\ [wac] & [wbc] & \ldots & [wfc] \\ \cdot & \cdot & \cdot & \cdot \\ [waf] & [wbf] & \ldots & [wff] \end{vmatrix}}.$$

As this is the value of x obtained from the ordinary normal equations of the Method of Least Squares, we see that the present investigation leads to the establishment of that Method.

116. **The Weight of a Linear Function.**—It remains to show how the equation

$$\frac{1}{W} = \frac{\lambda_1^2}{w_1} + \frac{\lambda_2^2}{w_2} + \ldots + \frac{\lambda_s^2}{w_s} \qquad \text{(E)}$$

has been obtained by the different writers who have furnished proofs of this type. Laplace obtained it by investigating the value of a linear sum of errors

$$\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s$$

when each of the errors has a definite law of facility, say the probability that the rth error lies between ϵ_r and $\epsilon_r + d\epsilon_r$ is $\phi(\epsilon_r)d\epsilon_r$. This investigation we have given in a form somewhat different to Laplace's, in § 86: as is there shown, it leads to the result that if

$$\int_0^\infty x^2\phi(x)\,dx$$

is denoted by k^2, then the probability that the linear sum $\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s$ lies between $-l$ and $+l$ is, when s is large and under certain conditions,

$$\frac{1}{(\pi\Sigma\lambda^2k^2)^{\frac{1}{2}}}\int_0^l e^{-\frac{u^2}{4\Sigma\lambda^2k^2}}du,$$

and whatever l may be, this is a maximum when

$$\lambda_1^2k_1^2 + \lambda_2^2k_2^2 + \ldots + \lambda_s^2k_s^2$$

is a minimum. Defining the weights as inversely proportional to $k_1^2, \ldots, k_s^2$, we obtain the equation (E). It will be noticed that Laplace's method requires that the number of observations should be very large, or else that the elementary errors should follow the normal law of facility.

Gauss, on the other hand, writing

$$\begin{cases} \epsilon_1 = a_1x + b_1y + \ldots - n_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \epsilon_s = a_sx + b_sy + \ldots - n_s, \end{cases}$$

obtained accurately

$$x = (\lambda_1n_1 + \ldots + \lambda_sn_s) + (\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s),$$

where as before the λ's are multipliers which satisfy the equations

$$\begin{cases} \lambda_1a_1 + \ldots + \lambda_sa_s = 1, \\ \lambda_1b_1 + \ldots + \lambda_sb_s = 0, \\ \text{etc.,} \end{cases}$$

and therefore reduce the coefficient of x to unity and the coefficients of $y, z, \ldots$ to zero. He then assumed that the importance of the error $\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s$, *i.e.* the detriment of which it is the cause, may be represented by $(\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s)^2$. That is, $(\lambda_1\epsilon_1 + \ldots + \lambda_s\epsilon_s)^2$ is a function whose mean value is to be made a minimum. Thus instead of securing that the probability of a zero error is to be a maximum, as he did in the *Theoria Motus* proof, Gauss now endeavoured to diminish for each unknown quantity, the probable value of the square of the error committed. Now

$$(\lambda_1\epsilon_1 + . \ . \ . + \lambda_s\epsilon_s)^2 = \Sigma\lambda_1^2\epsilon_1^2 + 2\Sigma\lambda_1\lambda_2\epsilon_1\epsilon_2.$$

The mean value of ϵ_1^2 is

$$\int_{-\infty}^{\infty} \epsilon^2\phi_1(\epsilon)\,d\epsilon = 2k_1^2,$$

and the mean value of $\epsilon_1\epsilon_2$ is zero.

Hence the quantity to be made a minimum is

$$\lambda_1^2k_1^2 + . \ . \ . + \lambda_s^2k_s^2$$

as in Laplace's proof: and it appears from Gauss's proof that this formula can be obtained by making minimum the mean square of error (which is the average of the true square of error for an infinite number of cases); or in other words the Method of Least Squares gives a result such that, if the whole system of observations were repeated an infinite number of times, the average value of the square of the error would be a minimum. The postulate of the arithmetic mean, on which the *Theoria Motus* proof was based, is not needed here. Gauss himself decidedly preferred this proof to his earlier treatment.*

Some modern writers have derived the equation (E) directly from assumptions regarding *weight*, which is taken as a fundamental notion, and have thereby succeeded in establishing the Method of Least Squares without any appeal to the ordinary theories of probability of error. This may be done in the following way: †

Let n be an estimated value of a quantity x: then we shall associate with this estimate a number w which will be called its *weight*, and *we shall assume as an axiom* that the weight of the estimate λn of the value of λx (where λ is any number) is of the form $wf(\lambda)$, where $f(\lambda)$ is some function of λ. If μ is any number, the weight of the estimate $\mu\lambda n$ of the value of $\mu\lambda x$ is therefore $wf(\lambda)f(\mu)$: but it is also $f(\lambda\mu)$; so the function f must satisfy the equation

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* Cf. letter of Gauss to Schumacher, November 25, 1844, in Gauss, *Werke*, 8, p. 147.

† Cf. Bernstein and Baer, *Math. Ann.* 76 (1915), p. 284.

$$f(\lambda\mu)=f(\lambda)f(\mu),$$

whence we have

$$f(\lambda)=\lambda^k,$$

where k is some number. Thus *if the weight of the estimate* n *for* x *be* w, *the weight of the estimate* λn *for* λx *is*

$$w\lambda^k. \tag{1}$$

Next suppose an estimate l for x has the weight p and an independent estimate m for y has the weight q, and let the weight of the estimate $l+m$ for $x+y$ be r.

We shall assume as an axiom that r is given by an equation of the form

$$\psi(r)=\psi(p)+\psi(q), \tag{2}$$

this being in fact the definition of independence of the two estimates. Since the estimate λl for λx has the weight $p\lambda^k$ etc., we have

$$\psi(r\lambda^k)=\psi(p\lambda^k)+\psi(q\lambda^k). \tag{3}$$

We have therefore to find the nature of the function ψ for which equation (3) is a consequence of equation (2), whatever λ may be. Regarding r as a function of p and q, we have by differentiating (2) with respect to p

$$\psi'(r)\frac{\partial r}{\partial p}=\psi'(p),$$

and by differentiating (3)

$$\psi'(r\lambda^k)\frac{\partial r}{\partial p}=\psi'(p\lambda^k).$$

Thus

$$\frac{\psi'(p\lambda^k)}{\psi'(p)}=\frac{\psi'(r\lambda^k)}{\psi'(r)},$$

and similarly the latter fraction is equal to

$$\frac{\psi'(q\lambda^k)}{\psi'(q)}.$$

Therefore $\dfrac{\psi'(p\lambda^k)}{\psi'(p)}$ is independent of p, and therefore since $p\lambda^k$ is symmetrical with respect to p and λ^k, we must have

$$\frac{\psi'(p\lambda^k)}{\psi'(p)\psi'(\lambda^k)},$$

a constant independent of p and λ^k. The function ψ' therefore satisfies the functional equation

$$g(x, y) = cg(x)g(y),$$

which shows that $\psi'(x)$ is a mere power of x multiplied by a constant, and therefore $\psi(x)$ is also a power of x multiplied by a constant. This constant has no influence on equation (2), and we can therefore write without loss of generality

$$\psi(x) = x^\gamma. \tag{4}$$

Next let n_1 and n_2 be two independent estimates of x, each of weight 1. Then by (1), $\frac{1}{2}n_1$ and $\frac{1}{2}n_2$ are two independent estimates of $\frac{1}{2}x$, each of weight $\frac{1}{2^k}$: and therefore by (2) and (4), $\frac{1}{2}n_1 + \frac{1}{2}n_2$ is an estimate of x of weight r, where

$$r^\gamma = \left(\frac{1}{2^k}\right)^\gamma + \left(\frac{1}{2^k}\right)^\gamma = \frac{1}{2^{k\gamma - 1}}.$$

We shall now assume as an axiom that if two independent estimates of the same quantity are each of unit weight, their arithmetic mean is an estimate of the same quantity having the weight 2. We have therefore $r = 2$, and consequently

$$2^\gamma = \frac{1}{2^{k\gamma-1}}$$

or

$$(k+1)\gamma = 1. \tag{5}$$

Combining (1), (2), (4), (5) we see that if n_1 is an estimate of x_1 with weight w_1, and if n_2 is an independent estimate of x_2 with weight w_2, then $\lambda_1 n_1 + \lambda_2 n_2$ is an estimate of $\lambda_1 x_1 + \lambda_2 x_2$ with the weight r, where

$$r^\gamma = w_1{}^\gamma \lambda_1{}^{1-\gamma} + w_2{}^\gamma \lambda_2{}^{1-\gamma}. \tag{6}$$

Now suppose that $n_1, n_2, \ldots, n_s$ are estimates of $x_1, x_2, \ldots, x_s$ respectively, each of weight 1. Then by repeated applications of (6), we see that $n_1 + n_2 + \ldots + n_s$ is an estimate of $x_1 + x_2 + \ldots + x_s$ with the weight r where

$$r^\gamma = 1^\gamma + 1^\gamma + \ldots + 1^\gamma = s,$$

so

$$r = s^{\frac{1}{\gamma}}.$$

Now *we shall assume as an axiom* that when the weights of $n_1, \ldots, n_s$ are each unity, the weight of $n_1 + n_2 + \ldots + n_s$ is of the order $\frac{1}{s}$, *i.e.* the product of this weight and s is always finite and bounded as $s \to \infty$. Therefore

$$\frac{1}{\gamma} = -1 \text{ or } \gamma = -1. \tag{7}$$

By (6) repeated and (7), we see that *if* $n_1, n_2, \ldots, n_s$ *are estimates of* $x_1, x_2, \ldots, x_s$ *respectively, of weights* $w_1, w_2, \ldots, w_s$, *then* $\lambda_1 n_1 + \lambda_2 n_2 + \ldots + \lambda_s n_s$ *is an estimate of* $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_s x_s$ *with the weight* W, where

$$\frac{1}{W} = \frac{\lambda_1^2}{w_1} + \frac{\lambda_2^2}{w_2} + \ldots + \frac{\lambda_s^2}{w_s}.$$

This is equation (E), from which, as we have seen, the Method of Least Squares may be derived.

117. **Solution of the Normal Equations.**—We shall now discuss the solution of the normal equations, which are equal in number to the number m of unknowns, and which we shall write

$$\begin{cases} a_{11}x + a_{12}y + \ldots + a_{1m}t = c_1, \\ a_{21}x + a_{22}y + \ldots + a_{2m}t = c_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x + a_{m2}y + \ldots + a_{mm}t = c_m, \end{cases}$$

where $a_{qp} = a_{pq}$.

Let D denote the determinant $\|a_{pq}\|$, and let A_{pq} denote the cofactor of a_{pq} in D. Then the solution of the above equations is known to be

$$\begin{cases} Dx = A_{11}c_1 + A_{21}c_2 + \ldots + A_{m1}c_m, \\ Dy = A_{12}c_1 + A_{22}c_2 + \ldots + A_{m2}c_m, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Dt = A_{1m}c_1 + A_{2m}c_2 + \ldots + A_{mm}c_m. \end{cases}$$

The problem is therefore to calculate D and its minors A_{pq}. To effect this, by the repeated application of the theorem of § 38, we reduce D to a determinant of lower order: the process may conveniently be stopped when the reduced determinant is of the 4th order, so that we have, say,

$$D = M \begin{vmatrix} b_{m_1n_1} & b_{m_1n_2} & b_{m_1n_3} & b_{m_1n_4} \\ b_{m_2n_1} & b_{m_2n_2} & b_{m_2n_3} & b_{m_2n_4} \\ b_{m_3n_1} & b_{m_3n_2} & b_{m_3n_3} & b_{m_3n_4} \\ b_{m_4n_1} & b_{m_4n_2} & b_{m_4n_3} & b_{m_4n_4} \end{vmatrix},$$

where M denotes an external factor, and each element b_{pq} has been derived from an original element a_{pq} by a succession of processes of the kind described in § 38.

Now if b_{pq} is one of these 16 surviving elements, the minor A_{pq} of D may be reduced, by precisely the same transformations as D, to the product of M and a determinant of the 3rd order: indeed this reduced form of A_{pq} may be derived from the above reduced form of D by merely forming the cofactor of b_{pq} in it. Thus a single reduction-process furnishes not only D, but also 16 of the minors A_{pq}.

Taking for example the case where we have 6 unknowns, so that D is of the 6th order, we may reduce D by taking a_{11} and b_{22} in succession as the pivotal elements, and shall thereby obtain the minors

$$A_{33}\ A^*_{34}\ A^*_{35}\ A^*_{36}\ A_{44}\ A^*_{45}\ A^*_{46}\ A_{55}\ A^*_{56}\ A_{66},$$

those with an asterisk being obtained twice.

We can next reduce D independently by taking a_{34} and b_{56} in succession as the pivotal elements, and shall thereby obtain the minors

$$A_{11}A^*_{12}\ A_{13}\ A_{15}\ A_{22}\ A_{23}\ A_{25}\ A_{41}\ A_{42}\ A_{43}\ A_{45}\ A_{61}\ A_{62}\ A_{63}\ A_{65};$$

so that altogether of the minors

$$\begin{array}{cccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ & & A_{33} & A_{34} & A_{35} & A_{36} \\ & & & A_{44} & A_{45} & A_{46} \\ & & & & A_{55} & A_{56} \\ & & & & & A_{66} \end{array}$$

we obtain fourteen of them once, three of them twice, and four three times. These multiple determinations serve as checks to the calculations.

Ex. 1.—*To solve the equations*

$$\begin{aligned} x+3y-2z\ -2v &= 0\cdot 5, \\ 3x+4y-5z+\ u-3v &= 5\cdot 4, \\ -2x-5y+3z-2u+2v &= -5\cdot 0, \\ y-2z+5u+3v &= 7\cdot 5, \\ -2x-3y+2z+3u+4v &= 3\cdot 3, \end{aligned}$$

we first form the determinant

$$D=\begin{vmatrix} 1 & 3 & -2 & 0 & -2 \\ 3 & 4 & -5 & 1 & -3 \\ -2 & -5 & 3 & -2 & 2 \\ 0 & 1 & -2 & 5 & 3 \\ -2 & -3 & 2 & 3 & 4 \end{vmatrix}$$

and taking a_{11} as the pivotal element, we have at once

$$D=D'\equiv\begin{vmatrix} -5 & 1 & 1 & 3 \\ 1 & -1 & -2 & -2 \\ 1 & -2 & 5 & 3 \\ 3 & -2 & 3 & 0 \end{vmatrix}=-25.$$

We now find the co-factors of the elements a_{pq} of D, corresponding to the surviving elements in D′. For example A_{23}, the co-factor of -5 in D, is evidently (§ 38) the co-factor of $b_{12}=1$ in D′, so

$$A_{23} = -\begin{vmatrix} 1 & -2 & -2 \\ 1 & 5 & 3 \\ 3 & 3 & 0 \end{vmatrix} = 3.$$

In this way we find the values

$$\begin{array}{llll} A_{22} = 13 & A_{32} = 3 & A_{42} = -11 & A_{52} = 16 \\ A_{23} = 3 & A_{33} = 18 & A_{43} = 9 & A_{53} = -4 \\ A_{24} = -11 & A_{34} = 9 & A_{44} = 17 & A_{54} = -27 \\ A_{25} = 16 & A_{35} = -4 & A_{45} = -27 & A_{55} = 37. \end{array}$$

Now selecting a_{24} as pivotal element, we form the determinant

$$D = D'' = \begin{vmatrix} 1 & 3 & -2 & -2 \\ 4 & 3 & -7 & -4 \\ -15 & -19 & 23 & 18 \\ -11 & -15 & 17 & 13 \end{vmatrix} = -25$$

and then determine the co-factors of the elements a_{pq} of D, corresponding to c_{pq} of D″. For example,

$$A_{11} = \begin{vmatrix} 3 & -7 & -4 \\ -19 & 23 & 18 \\ -15 & 17 & 13 \end{vmatrix} = 52.$$

In this way we find the values

$$\begin{array}{llll} A_{21} = -1, & A_{31} = 19, & A_{41} = -3, & A_{51} = 18, \\ A_{11} = 52, & A_{12} = -1, & A_{13} = 19, & A_{15} = 18. \end{array}$$

Now taking a_{42} as pivotal element in D we obtain the remaining co-factors:

$$A_{21} = -1, \quad A_{14} = -3.$$

The solution of the above equations can be written down at once:

$$\begin{aligned} Dx &= 0{\cdot}5A_{11} + 5{\cdot}4A_{21} - 5{\cdot}0A_{31} + 7{\cdot}5A_{41} + 3{\cdot}3A_{51} \\ &= 26{\cdot}0 - 5{\cdot}4 - 95{\cdot}0 - 22{\cdot}5 + 59{\cdot}4 \end{aligned}$$

or $-25x = -37{\cdot}5$; whence $x = 1{\cdot}5$.

$$\begin{aligned} Dy &= 0{\cdot}5A_{12} + 5{\cdot}4A_{22} - 5{\cdot}0A_{32} + 7{\cdot}5A_{42} + 3{\cdot}3A_{52} \\ &= -0{\cdot}5 + 70{\cdot}2 - 15{\cdot}0 - 82{\cdot}5 + 52{\cdot}8 \end{aligned}$$

or $-25y = 25{\cdot}0$; whence $y = -1{\cdot}0$.

$$\begin{aligned} Dz &= 0{\cdot}5A_{13} + 5{\cdot}4A_{23} - 5{\cdot}0A_{33} + 7{\cdot}5A_{43} + 3{\cdot}3A_{53} \\ &= 9{\cdot}5 + 16{\cdot}2 - 90{\cdot}0 + 67{\cdot}5 - 13{\cdot}2 \end{aligned}$$

or $-25z = -10{\cdot}0$; whence $z = 0{\cdot}4$.

$$\begin{aligned} Du &= 0{\cdot}5A_{14} + 5{\cdot}4A_{24} - 5{\cdot}0A_{34} + 7{\cdot}5A_{44} + 3{\cdot}3A_{54} \\ &= -1{\cdot}5 - 59{\cdot}4 - 45{\cdot}0 + 127{\cdot}5 - 89{\cdot}1 \end{aligned}$$

or $-25u = -67{\cdot}5$; whence $u = 2{\cdot}7$.

$$\begin{aligned} Dv &= 0{\cdot}5A_{15} + 5{\cdot}4A_{25} - 5{\cdot}0A_{35} + 7{\cdot}5A_{45} + 3{\cdot}3A_{55} \\ &= 9{\cdot}0 + 86{\cdot}4 + 20{\cdot}0 - 202{\cdot}5 + 122{\cdot}1 \end{aligned}$$

or $-25v = 35{\cdot}0$; whence $v = -1{\cdot}4$.

Ex. 2.—*Solve the equations*

$$\begin{array}{rcl} 5x-3y+7z+u+2v &=& 356\cdot 8, \\ -3x+2y+2u+2v &=& 60\cdot 5, \\ 7x+2z+u-v &=& 167\cdot 0, \\ x+2y+z+u-5v &=& -\ 7\cdot 5, \\ 2x+2y-z-5u+v &=& -\ 71\cdot 2. \end{array}$$

118. Final Control of the Calculations.—When the most plausible values $x_0, y_0, \ldots, t_0$ have been found by solution of the normal equations, we can calculate the *residuals* $v_1, \ldots, v_s$ defined by the equations

$$\left.\begin{array}{c} a_1x_0+b_1y_0+\ldots+f_1t_0-n_1=v_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx_0+b_sy_0+\ldots+f_st_0-n_s=v_s \end{array}\right\}. \qquad (1)$$

These residuals v are (as will appear later) required in order to determine the mean error of our results. Meanwhile we shall show how they may be used to furnish a check on the working hitherto.

We have

$$\begin{aligned} [v^2] &= (a_1x_0+\ldots+f_1t_0-n_1)^2+\ldots+(a_sx_0+\ldots+f_st_0-n_s)^2 \\ &= [aa]x_0^2+[bb]y_0^2+\ldots+2[ab]x_0y_0+\ldots-2[an]x_0-\ldots+[nn] \\ &= x_0\{[aa]x_0+\ldots+[af]t_0-[an]\}+y_0\{[ab]x_0+\ldots+[bf]t_0-[bn]\} \\ &\qquad +\ldots-[an]x_0-[bn]y_0-\ldots-[fn]t_0+[nn], \end{aligned}$$

and by virtue of the normal equations this expression reduces to the last set of terms, so we have

$$[v^2]=-[an]x_0-[bn]y_0-\ldots-[fn]t_0+[nn].$$

This equation may be used as a control for checking the accuracy of the whole set of computations, $[v^2]$ being computed directly by squaring and adding the residuals. We have previously (§§ 108, 117) described controls on the formation of the normal equations and on the calculation of the determinants involved in their solution.

119. Gauss's Method of Solution of the Normal Equations.—The method of solving normal equations given by Gauss[*] differs, in form at any rate, from the determinantal method described in § 117. It may be described thus:

From the normal equation in x, we find x in terms of the other variables and substitute this value in the remaining normal equations:

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[*] *Theoria Combinationis, Supplementum.* The method is fully described by Encke, *Berlin. astronomische Jahrbuch* (1835), pp. 267, 272, and (1836), p. 263.

this gives us the "first transformed system," which involves only $y, z, \ldots, t$, and is (like the original system) axisymmetric.

From the first equation of the first transformed system (viz. that equation which was derived from the normal equation in y) we find y in terms of $z, \ldots, t$, and substitute this value in the remaining equations of the first transformed system: this gives us the "second transformed system," which involves only $z, \ldots, t$, and is also axisymmetric. Proceeding in this way we obtain at last an equation which involves only t and from which t can therefore be determined: we then determine all the other unknowns, each from the equation which was used for its elimination.

Thus if the original normal equations are

$$\begin{aligned} ax + hy + gz &= l, \\ hx + by + fz &= m, \\ gx + fy + cz &= n, \end{aligned}$$

the set of equations which are used in Gauss's method for the final determination of x, y, z are

$$\left.\begin{aligned} &ax + hy + gz = l \\ &\begin{vmatrix} a & h \\ h & b \end{vmatrix} y + \begin{vmatrix} a & g \\ h & f \end{vmatrix} z = \begin{vmatrix} a & l \\ h & m \end{vmatrix} \\ &\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} z = \begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix} \end{aligned}\right\}. \qquad (1)$$

Now if T denote the quadratic form which represents the sum of the squares of the errors, namely

$$\mathrm{T} \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 2lx - 2my - 2nz + u,$$

it is known that T can be expressed as a sum of squares in the form

$$\begin{aligned} \mathrm{T} = {} & \frac{1}{a}(ax + hy + gz - l)^2 + \frac{1}{a\begin{vmatrix} a & h \\ h & b \end{vmatrix}} \left\{ \begin{vmatrix} a & h \\ h & b \end{vmatrix} y + \begin{vmatrix} a & g \\ h & f \end{vmatrix} z - \begin{vmatrix} a & l \\ h & m \end{vmatrix} \right\}^2 \\ & + \frac{1}{\begin{vmatrix} a & h \\ h & b \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}} \left\{ \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} z - \begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix} \right\}^2 + \frac{1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}} \cdot \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & u \end{vmatrix} \end{aligned}$$

which shows at once that the equations (1) must necessarily represent the conditions that T is to be a minimum, and shows also that Gauss's method of solution is substantially equivalent to the reduction of a quadratic form to a sum of squares.

Ex. 1.—*To solve the equations*

$$\begin{aligned} x + 3y - 2z - \phantom{2u + {}} 2v &= 0{\cdot}5, & (1) \\ 3x + 4y - 5z + u - 3v &= 5{\cdot}4, & (2) \\ -2x - 5y + 3z - 2u + 2v &= -5{\cdot}0, & (3) \\ y - 2z + 5u + 3v &= 7{\cdot}5, & (4) \\ -2x - 3y + 2z + 3u + 4v &= 3{\cdot}3. & (5) \end{aligned}$$

Eliminating x from equations (1) and (2), we have

$$\begin{vmatrix}1 & 3\\ 3 & 4\end{vmatrix} y + \begin{vmatrix}1 & -2\\ 3 & -5\end{vmatrix} z + \begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix} u + \begin{vmatrix}1 & -2\\ 3 & -3\end{vmatrix} v = \begin{vmatrix}1 & 0{\cdot}5\\ 3 & 5{\cdot}4\end{vmatrix}$$

$$-5y + z + u + 3v = 3{\cdot}9, \tag{6}$$

and similarly, by combining (1) with (3), (4), (5) in succession, we obtain the equations

$$\begin{aligned} y - z - 2u - 2v &= -4{\cdot}0, & (7)\\ y - 2z + 5u + 3v &= 7{\cdot}5, & (8)\\ 3y - 2z + 3u &= 4{\cdot}3. & (9)\end{aligned}$$

Now eliminating y from (6) and (7), we have

$$\begin{vmatrix}-5 & 1\\ 1 & -1\end{vmatrix} z + \begin{vmatrix}-5 & 1\\ 1 & -2\end{vmatrix} u + \begin{vmatrix}-5 & 3\\ 1 & -2\end{vmatrix} v = \begin{vmatrix}-5 & 3{\cdot}9\\ 1 & -4{\cdot}0\end{vmatrix}$$

or

$$4z + 9u + 7v = 16{\cdot}1, \tag{10}$$

and similarly, combining (6) with (8) and (9) in succession, we obtain

$$\begin{aligned} 9z - 26u - 18v &= -41{\cdot}4, & (11)\\ 7z - 18u - 9v &= -33{\cdot}2. & (12)\end{aligned}$$

Continuing this process of elimination, the following equations are obtained:

$$\begin{aligned} 185u + 135v &= 310{\cdot}5, & (13)\\ 135u + 85v &= 245{\cdot}5, & (14)\end{aligned}$$

and finally, from (14), (13), (10), (6), (1), we determine the results $v = -1{\cdot}4$, $u = 2{\cdot}7$, $z = 0{\cdot}4$, $y = -1{\cdot}0$, and $x = 1{\cdot}5$.

Ex. 2.—*Solve by Gauss's method the equations*

$$\begin{aligned} 5x - 3y + 7z + u + 2v &= 356{\cdot}8,\\ -3x + 2y + 2u + 2v &= 60{\cdot}5,\\ 7x + 2z + u - v &= 167{\cdot}0,\\ x + 2y + z + u - 5v &= -7{\cdot}5,\\ 2x + 2y - z - 5u + v &= -71{\cdot}2.\end{aligned}$$

120. The "Method of Equal Coefficients" for the Solution of Linear Equations.[*]—In a method of performing the elimination, which is known as the *Method of Equal Coefficients*, the two first equations are reduced to two equations between *each* of the two first unknowns and the remaining unknowns; then these two with the third equation of the original system are reduced to three equations between *each* of the three first unknowns and the remaining unknowns, and so on. The advantage of this method is that the same coefficients occur repeatedly, the elimination being thereby greatly facilitated. The method involves certain divisions, which, however, *should always give exact quotients without remainders,* thereby constituting a control of the accuracy of the computation.

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[*] B. I. Clasen, *Brux. Soc. Sc.* **12** (1888), A 50-59, B 251-281.

Thus let the equations be

$$
\begin{aligned}
a_1x + b_1y + c_1z + d_1u + e_1v &= f_1, & \mathbf{1}_1\\
a_2x + b_2y + c_2z + d_2u + e_2v &= f_2, & \mathbf{1}_2\\
a_3x + b_3y + c_3z + d_3u + e_3v &= f_3, & \mathbf{1}_3\\
a_4x + b_4y + c_4z + d_4u + e_4v &= f_4, & \mathbf{1}_4\\
a_5x + b_5y + c_5z + d_5u + e_5v &= f_5. & \mathbf{1}_5
\end{aligned}
$$

Eliminating x between $\mathbf{1}_1$ and $\mathbf{1}_2$, we have

$$|a_1b_2|\,y + |a_1c_2|\,z + |a_1d_2|\,u + |a_1e_2|\,v = |a_1f_2|, \qquad \mathbf{2}_2$$

and eliminating y between $\mathbf{1}_1$ and $\mathbf{2}_2$ by forming the sum of $-|a_1b_2|\,\mathbf{1}_1$ and $b_1\mathbf{2}_2$ and dividing by a_1, we have

$$-|a_1b_2|\,x + |b_1c_2|\,z + |b_1d_2|\,u + |b_1e_2|\,v = |b_1f_2|. \qquad \mathbf{2}_1$$

Now multiplying $\mathbf{2}_1$ by a_3, $\mathbf{2}_2$ by $-b_3$, $\mathbf{1}_3$ by $|a_1b_2|$ and adding, we have

$$|a_1b_2c_3|\,z + |a_1b_2d_3|\,u + |a_1b_2e_3|\,v = |a_1b_2f_3|. \qquad \mathbf{3}_3$$

Eliminating z between $\mathbf{2}_2$ and $\mathbf{3}_3$ by forming the combination $|a_1b_2c_3|\,\mathbf{2}_2 - |a_1c_2|\,\mathbf{3}_3$, and dividing by $-|a_1b_2|$, we have

$$-|a_1b_2c_3|\,y + |a_1c_2d_3|\,u + |a_1c_2e_3|\,v = |a_1c_2f_3|, \qquad \mathbf{3}_2$$

and eliminating z between $\mathbf{2}_1$ and $\mathbf{3}_3$ by forming the combination $|a_1b_2c_3|\,\mathbf{2}_1 - |b_1c_2|\,\mathbf{3}_3$ and dividing by $-|a_1b_2|$, we have

$$|a_1b_2c_3|\,x + |b_1c_2d_3|\,u + |b_1c_2e_3|\,v = |b_1c_2f_3|. \qquad \mathbf{3}_1$$

Now form the combination $-a_4\mathbf{3}_1 + b_4\mathbf{3}_2 - c_4\mathbf{3}_3 + |a_1b_2c_3|\,\mathbf{1}_4$. We get

$$|a_1b_2c_3d_4|\,u + |a_1b_2c_3e_4|\,v = |a_1b_2c_3f_4|. \qquad \mathbf{4}_4$$

Eliminating u between this equation and $\mathbf{3}_3$ by forming $|a_1b_2c_3d_4|\,\mathbf{3}_3 - |a_1b_2d_3|\,\mathbf{4}_4$ and dividing by $-|a_1b_2c_3|$, we have

$$-|a_1b_2c_3d_4|\,z + |a_1b_2d_3e_4|\,v = |a_1b_2d_3f_4|, \qquad \mathbf{4}_3$$

and similarly

$$|a_1b_2c_3d_4|\,y+|a_1c_2d_3e_4|\,v=|a_1c_2d_3f_4|, \qquad \mathbf{4}_2$$

$$-|a_1b_2c_3d_4|\,x+|b_1c_2d_3e_4|\,v=|b_1c_2d_3f_4|. \qquad \mathbf{4}_1$$

Now form the combination $a_5\mathbf{4}_1-b_5\mathbf{4}_2+c_5\mathbf{4}_3-d_5\mathbf{4}_4+|a_1b_2c_3d_4|\mathbf{1}_5$. We get

$$|a_1b_2c_3d_4e_5|\,v=|a_1b_2c_3d_4f_5|. \qquad \mathbf{5}_5$$

Eliminating v between $\mathbf{5}_5$ and $\mathbf{4}_4$ by forming $|a_1b_2c_3d_4e_5|\,\mathbf{4}_4-|a_1b_2c_3e_4|\,\mathbf{5}_5$ and dividing by $|a_1b_2c_3d_4|$, we have

$$-|a_1b_2c_3d_4e_5|\,u=|a_1b_2c_3e_4f_5|, \qquad \mathbf{5}_4$$

and similarly

$$|a_1b_2c_3d_4e_5|\,z=|a_1b_2d_3e_4f_5|, \qquad \mathbf{5}_3$$

$$-|a_1b_2c_3d_4e_5|\,y=|a_1c_2d_3e_4f_5|, \qquad \mathbf{5}_2$$

$$|a_1b_2c_3d_4e_5|\,x=|b_1c_2d_3e_4f_5|. \qquad \mathbf{5}_1$$

The solution is thus completed.

Ex. 1.—*To solve the equations*

$$x+3y-2z\qquad-2v=0{\cdot}5, \qquad \mathbf{1}_1$$

$$3x+4y-5z+u-3v=5{\cdot}4, \qquad \mathbf{1}_2$$

$$-2x-5y+3z-2u+2v=-5{\cdot}0, \qquad \mathbf{1}_3$$

$$y-2z+5u+3v=7{\cdot}5, \qquad \mathbf{1}_4$$

$$-2x-3y+2z+3u+4v=3{\cdot}3. \qquad \mathbf{1}_5$$

Eliminating x from equations $\mathbf{1}_1$ and $\mathbf{1}_2$, we have

$$-5y+z+u+3v=3{\cdot}9, \qquad \mathbf{2}_2$$

and eliminating y between $\mathbf{1}_1$ and $\mathbf{2}_2$ by forming the sum of $5.\mathbf{1}_1$ and $3.\mathbf{2}_2$, we obtain

$$5x-7z+3u-v=14{\cdot}2, \qquad \mathbf{2}_1$$

in which the coefficient of x is minus the coefficient of y in $\mathbf{2}_2$.

Now multiply $\mathbf{2}_1$ by -2, $\mathbf{2}_2$ by 5, $\mathbf{1}_3$ by -5, and add to form the equation

$$4z+9u+7v=16{\cdot}1, \qquad \mathbf{3}_3$$

in which there are no terms in x and y.

Eliminating z between $\mathbf{3}_3$ and $\mathbf{2}_2$ by forming the combination $\frac{1}{5}(4.\mathbf{2}_2-\mathbf{3}_3)$, we obtain the equation

$$-4y-u+v=-0{\cdot}1, \qquad \mathbf{3}_2$$

in which the coefficient of y is minus the coefficient of z in $\mathbf{3}_3$.

Now eliminating z between $\mathbf{2}_1$ and $\mathbf{3}_3$ by forming the combination $\frac{1}{5}(4.\mathbf{2}_1+7.\mathbf{3}_3)$, we find

$$4x+15u+9v=33{\cdot}9, \qquad \mathbf{3}_1$$

in which equation the coefficient of x differs only in sign from the coefficient of y in 3_2. Now form the sum $3_2 + 2.3_3 + 4.1_4$,

$$37u + 27v = 62{\cdot}1, \tag{4_4}$$

and eliminate u between this equation and 3_3,

$$-37z - 4v = -9{\cdot}2. \tag{4_3}$$

The remaining equations of the process are

$$37y - 16v = -14{\cdot}6, \tag{4_2}$$

$$-37x + 18v = -80{\cdot}7, \tag{4_1}$$

$$-25v = 35{\cdot}0, \tag{5_5}$$

$$25u = 67{\cdot}5, \tag{5_4}$$

$$-25z = 10{\cdot}0, \tag{5_3}$$

$$25y = -25{\cdot}0, \tag{5_2}$$

$$-25x = -37{\cdot}5. \tag{5_1}$$

The required solution is therefore

$$x = 1{\cdot}5, \quad y = -1{\cdot}0, \quad z = 0{\cdot}4, \quad u = 2{\cdot}7, \quad v = -1{\cdot}4.$$

Ex. 2.—*Solve by the above method the equations*

$$\begin{array}{rcl}
5x - 3y + 7z + u + 2v &=& 356{\cdot}8, \\
-3x + 2y + 2u + 2v &=& 60{\cdot}5, \\
7x + 2z + u - v &=& 167{\cdot}0, \\
x + 2y + z + u - 5v &=& -\ 7{\cdot}5, \\
2x + 2y - z - 5u + v &=& -\ 71{\cdot}2.
\end{array}$$

121. **Comparison of the Three Methods of solving Normal Equations.**—Comparing the three methods which have been given—the Determinantal method (§ 117), Gauss's method (§ 119), and the method of Equal Coefficients (§ 120)—we may say that the Determinantal method is on the whole the best for the solution of a set of normal equations. It should be observed, however, that the superiority of the Determinantal method depends on the circumstance that a set of normal equations is always axisymmetric (*i.e.* the coefficient of y in the normal equation for x is equal to the coefficient of x in the normal equation for y). For a set of linear equations which is not axisymmetric, the Determinantal method is inferior to the method of Gauss.

122. **The Weight of the Unknowns.**—We shall now

investigate the weights of the determinations of the unknowns $x, y, \ldots, t$. Denote the most probable values of the unknowns by $x_0, y_0, \ldots, t_0$. We have in the notation of § 117

$$Dx_0 = A_{11}c_1 + A_{21}c_2 + \ldots + A_{m1}c_m,$$

where D and the minor A_{pq} depend only on the accurately-known coefficients $a, b, \ldots$ of the equations of condition and where

$$\begin{aligned} c_1 &= w_1a_1n_1 + w_2a_2n_2 + \ldots + w_sa_sn_s \\ c_2 &= w_1b_1n_1 + w_2b_2n_2 + \ldots + w_sb_sn_s \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

so

$$Dx_0 = \xi_1w_1n_1 + \xi_2w_2n_2 + \ \ldots + \xi_sw_sn_s, \tag{1}$$

where

$$\xi_1 = A_{11}a_1 + A_{21}b_1 + \ldots = \begin{vmatrix} a_1 & b_1 & c_1 & \ldots & f_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3m} \\ \cdot & & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mm} \end{vmatrix},$$

$$\xi_2 = \begin{vmatrix} a_2 & b_2 & c_2 & \ldots & f_2 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mm} \end{vmatrix}, \text{ etc.}$$

By giving to $n_1, \ldots, n_s$ the particular values $a_1, \ldots, a_s$, in which case we should evidently have $x_0 = 1$, we derive from equations (1) the result

$$D = [\xi wa], \tag{2}$$

and similarly by giving to $n_1, \ldots, n_s$ the values $b_1, \ldots, b_s$, in which case $x_0 = 0$, we have

$$0 = [\xi wb]. \tag{3}$$

Let w_x be the weight of the determination of x_0. Then since x_0 is given by equation (1), we have by the formula for the weight of a linear function of $n_1, \ldots, n_s$,

$$\frac{D^2}{w_x} = \xi_1^2 w_1 + \xi_2^2 w_2 + \ldots + \xi_s^2 w_s$$

$$= \begin{vmatrix} w_1\xi_1 a_1 & w_1\xi_1 b_1 & \ldots & w_1\xi_1 f_1 \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix} + \begin{vmatrix} w_2\xi_2 a_2 & w_2\xi_2 b_2 & \ldots & w_2\xi_2 f_2 \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix} + \ldots$$

$$+ \begin{vmatrix} w_s\xi_s a_s & w_s\xi_s b_s & \ldots & w_s\xi_s f_s \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix}$$

$$= \begin{vmatrix} [wa\xi] & [wb\xi] & \ldots & [wf\xi] \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix}$$

$$= \begin{vmatrix} D & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix} \text{ by equations (2) and (3).}$$

Therefore $$\frac{D^2}{w_x} = DA_{11}$$

or $$\frac{1}{w_x} = \frac{A_{11}}{D},$$

and similarly $$\frac{1}{w_y} = \frac{A_{22}}{D} \ldots$$

These formulae give the weights of the determination of $x, y, \ldots$; they are due to Gauss.*

It is a consideration in favour of the determinantal method of solution of the normal equations that as $D, A_{11}, A_{22}, \ldots$ are calculated in order to find $x, y, z, \ldots$, the method furnishes the weights without any fresh calculation.

Now let ϵ denote the mean error to be feared in an observation of unit weight, and let ϵ_x denote the mean error to be feared in the determination of x: then since $w_x = \frac{\epsilon^2}{\epsilon_x^2}$, we have

$$\epsilon_x = \epsilon\sqrt{\left(\frac{A_{11}}{D}\right)}.$$

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* *Theoria Combinationis*, § 21.

We shall see later (§ 124) how ϵ may be found in terms of the residuals of the equations of condition.

123. Weight of any Linear Function of Unknowns.—

Now let

$$u = l_1x + l_2y + \ldots + l_mt$$

be any linear function of the unknowns $x, y, z, \ldots, t$, so that its most probable value is

$$u_0 = l_1x_0 + l_2y_0 + \ldots + l_mt_0,$$

where $(x_0, y_0, \ldots, t_0)$ are the most probable values of the unknowns. Let it be required to find the weight of the equation $u = u_0$.*

It will be sufficient in the first place to take $u = lx + l'y$. Writing as before

$$\begin{cases} \xi_1 = A_{11}a_1 + A_{21}b_1 + \ldots + A_{m1}f_1, \\ \xi_2 = A_{11}a_2 + A_{21}b_2 + \ldots + A_{m1}f_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

and writing

$$\eta_1 = A_{12}a_1 + A_{22}b_1 + \ldots + A_{m2}f_1, \text{ etc.,}$$

we have

$$Dx_0 = \xi_1w_1n_1 + \xi_2w_2n_2 + \ldots + \xi_sw_sn_s,$$

$$Dy_0 = \eta_1w_1n_1 + \eta_2w_2n_2 + \ldots + \eta_sw_sn_s,$$

and therefore

$$Du_0 = (l\xi_1 + l'\eta_1)w_1n_1 + (l\xi_2 + l'\eta_2)w_2n_2 + \ldots + (l\xi_s + l'\eta_s)w_sn_s.$$

Therefore if w_u denote the weight of the equation $u = u_0$, we have

$$\frac{D^2}{w_u} = (l\xi_1 + l'\eta_1)^2w_1 + (l\xi_2 + l'\eta_2)^2w_2 + \ldots + (l\xi_s + l'\eta_s)^2w_s.$$

Now we have already proved that

$$\xi_1^2w_1 + \xi_2^2w_2 + \ldots + \xi_s^2w_s = DA_{11},$$

and in the same way we may show that

$$\xi_1\eta_1w_1 + \xi_2\eta_2w_2 + \ldots + \xi_s\eta_sw_s = DA_{12}.$$

Therefore

$$\frac{D^2}{w_u} = l^2DA_{11} + 2ll'DA_{12} + l'^2DA_{22}$$

or

$$\frac{D}{w_u} = A_{11}l^2 + 2A_{12}ll' + A_{22}l'^2,$$

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* Gauss, *Theoria Combinationis*, § 29.

and similarly in general if w_u is the weight of the function $l_1x + l_2y + \ldots + l_mt$, we have

$$\frac{D}{w_u} = A_{11}l_1^2 + A_{22}l_2^2 + \ldots + A_{mm}l_m^2 + 2A_{12}l_1l_2 + \ldots$$

This formula gives the weight of the determination of any linear function of the unknowns.

124. **The Mean Error of a Determination whose Weight is Unity.***—We shall now find the mean error of a determination whose weight is unity. For simplicity we shall suppose that the equations of condition have been multiplied by the square roots of their weights, so that they may be taken to be each of unit weight. Let them be as usual

$$\left.\begin{array}{l} a_1x + b_1y + \ldots + f_1t = n_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx + b_sy + \ldots + f_st = n_s \end{array}\right\}. \qquad (1)$$

Write

$a_rx + b_ry + \ldots + f_rt - n_r = E_r$; so $E_1, E_2, \ldots, E_s$ are the true errors of $n_1, \ldots, n_s$, if $x, y, \ldots, t$ are supposed to be the true values of the m unknowns.

Write also

$$a_rx_0 + b_ry_0 + \ldots + f_rt_0 - n_r = v_r;$$

so $v_1, \ldots, v_s$ are the residuals when the most probable values $x_0, \ldots, t_0$ are substituted in the equations of condition.

We have evidently

$$[av] = 0, \quad [bv] = 0, \ \ldots \ [fv] = 0. \qquad (2)$$

Also

$$\left.\begin{array}{l} E_1 - v_1 = a_1(x - x_0) + b_1(y - y_0) + \ldots + f_1(t - t_0) \\ \vdots \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ E_s - v_s = a_s(x - x_0) + b_s(y - y_0) + \ldots + f_s(t - t_0) \end{array}\right\}. \qquad (3)$$

Multiplying (3) by $v_1, \ldots, v_s$ and adding, remembering (2), we have

$$[Ev] - [vv] = 0.$$

Multiplying (3) by $E_1, \ldots, E_s$ and adding, we have

$$[EE] - [Ev] = [aE](x - x_0) + \ldots + [fE](t - t_0),$$

so
$$[EE] - [vv] = [aE](x - x_0) + \ldots + [fE](t - t_0). \qquad (4)$$

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* Gauss, *Theoria Combinationis*, §§ 37, 38, 39.

Now we have already seen that if coefficients $\xi_1, \ldots, \xi_s$ are defined by the equation

$$Dx_0 = \xi_1 n_1 + \xi_2 n_2 + \ldots + \xi_s n_s,$$

then $$[a\xi] = D, \quad [b\xi] = 0, \ldots, [f\xi] = 0,$$

and

$$[\epsilon\xi] = [a\xi]x_0 + [b\xi]y_0 + \ldots + [f\xi]t_0 - [n\xi] = Dx_0 - [n\xi] = 0.$$

Multiplying (3) by $\xi_1, \ldots, \xi_s$ respectively and adding, remembering the equations just written, we have

$$[\xi E] = (x - x_0)D.$$

Thus (4) becomes

$$D[EE] - D[vv] = [aE][\xi E] + [bE][\eta E] + \ldots + [fE][\tau E]. \quad (5)$$

This is a relation between the sum of the squares of the residuals and the sum of the squares of the true errors. As it stands, however, it is not sufficient to determine [EE], since the quantities E occur also on the right-hand side of the equation; but we may overcome this difficulty in the following way:

Suppose the set of observations repeated by the same observers with the same instruments under the same conditions, N times, where N is a great number. For each of these sets of observations we shall obtain an equation corresponding to (5). Let these equations be added together, and the resulting equation divided by N. The values of D and the a's, b's, . . ., f's, ξ's, . . ., τ's are the same for each set of observations, but the n's, E's, and v's differ from one set to another. Using the symbol Σ to denote summation over the different sets of observations, we have

$$\frac{D}{N}\Sigma[EE] - \frac{D}{N}\Sigma[vv]$$
$$= \frac{1}{N}\Sigma[aE][\xi E] + \frac{1}{N}\Sigma[bE][\eta E] + \ldots + \frac{1}{N}\Sigma[fE][\tau E].$$

Now if positive and negative errors are equally liable to occur, each of the sums $\frac{1}{N}\Sigma E_p E_q$ evidently tends to zero as N increases indefinitely: and then the equation becomes (when we understand the sign of equality now to indicate that the two sides of the equation differ only by negligible quantities)

$$\frac{\mathrm{D}}{\mathrm{N}}\Sigma[\mathrm{EE}] - \frac{\mathrm{D}}{\mathrm{N}}\Sigma[vv] = \frac{1}{\mathrm{N}}\Sigma(a_1\xi_1\mathrm{E}_1{}^2 + a_2\xi_2\mathrm{E}_2{}^2 + \ldots + a_s\xi_s\mathrm{E}_s{}^2) + \ldots$$
$$+ \frac{1}{\mathrm{N}}\Sigma(f_1\tau_1\mathrm{E}_1{}^2 + \ldots + f_s\tau_s\mathrm{E}_s{}^2).$$

Now $\frac{1}{\mathrm{N}}\Sigma\mathrm{E}_1{}^2 = \frac{1}{\mathrm{N}}\Sigma\mathrm{E}_2{}^2 = \ldots = \frac{1}{\mathrm{N}}\Sigma\mathrm{E}_s{}^2 =$ the square of the true quadratic mean error, *i.e.* the value which we should deduce from an indefinitely great number of observations.

Therefore, denoting the quadratic mean error by ϵ, we have

$$\begin{aligned}\mathrm{D}s\epsilon^2 - \frac{\mathrm{D}}{\mathrm{N}}\Sigma[vv] &\\ &= (a_1\xi_1 + a_2\xi_2 + \ldots + a_s\xi_s)\epsilon^2 + \ldots + (f_1\tau_1 + \ldots + f_s\tau_s)\epsilon^2 \\ &= \mathrm{D}m\epsilon^2,\end{aligned}$$

so

$$\epsilon^2 = \frac{1}{s-m}\cdot\frac{\Sigma[vv]}{\mathrm{N}}.$$

In this equation ϵ is the true value of the quadratic mean error to be feared in a determination whose weight is unity, and $\frac{1}{\mathrm{N}}\Sigma[vv]$ is the mean value of the sum of the squares of the residuals. Since the true value of $\frac{1}{\mathrm{N}}\Sigma[vv]$ is unknown, we are obliged to be content with that which is furnished by the unique actual set of observations: so for the most probable value of ϵ we have the equation

$$\epsilon^2 = \frac{[v^2]}{s-m}.$$

This is Gauss's expression for the quadratic mean error ϵ to be feared in a determination whose weight is unity; s denotes the number of equations of condition, and m is the number of the unknowns $x, y, \ldots, t$. The quantity

$$\sqrt{\left(\frac{s-m}{[v^2]}\right)}$$

is therefore well adapted to measure the precision of the given set of observations.

125. Evaluation of the Sum of the Squares of the Residuals.—We shall now show how $[v^2]$, the sum of the squares of the residuals, may be expressed in terms of the coefficients in the equations of condition and normal equations. We have

$$[v^2] = (a_1x_0 + b_1y_0 + \dots + f_1t_0 - n_1)^2 + \dots + (a_sx_0 + \dots + f_st_0 - n_s)^2$$
$$= [aa]x_0^2 + [bb]y_0^2 + \dots + 2[ab]x_0y_0 + \dots - 2[an]x_0 - \dots + [nn], \quad (1)$$

and we know that $[v^2]$ is the minimum value of the quadratic form

$$[aa]x^2 + [bb]y^2 + \dots + 2[ab]xy + \dots - 2[an]x \dots + [nn].$$

From the general theory of quadratic forms we know that the minimum value of this last form (which we know to be essentially positive, since it is a sum of squares) is

$$[v^2] = \frac{\begin{vmatrix} [aa] & [ab] & \dots & [af] & [an] \\ [ab] & [bb] & \dots & [bf] & [bn] \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [an] & [bn] & \dots & [fn] & [nn] \end{vmatrix}}{\begin{vmatrix} [aa] & [ab] & \dots & [af] \\ [ab] & [bb] & \dots & [bf] \\ \cdot & \cdot & \cdot & \cdot \\ [af] & [bf] & \dots & [ff] \end{vmatrix}}. \quad (2)$$

We may also establish this formula directly in the following way. We have already proved (§ 118) that

$$[v^2] = -[an]x_0 - [bn]y_0 - \dots - [fn]t_0 + [nn],$$

and substituting for $x_0, y_0, \dots, t_0$ their determinantal values, we obtain the equation (2).

Combining the results of this section with the equation (§ 124)

$$\epsilon^2 = \frac{[v^2]}{s - m}$$

and the equation (§ 122)

$$\epsilon_x^2 = \epsilon^2 \cdot \frac{A_{11}}{D},$$

we see that *the quadratic mean error to be feared in the determination of x is ϵ_x,* where

$$\epsilon_x^2 = \frac{\begin{vmatrix} [bb] & [bc] & \cdots & [bf] \\ [bc] & [cc] & \cdots & [cf] \\ \cdot & \cdot & \cdot & \cdot \\ [bf] & [cf] & \cdots & [ff] \end{vmatrix} \begin{vmatrix} [aa] & [ab] & \cdots & [af] & [an] \\ [ab] & [bb] & \cdots & [bf] & [bn] \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ [an] & [bn] & \cdots & [fn] & [nn] \end{vmatrix}}{(s-m) \begin{vmatrix} [aa] & [ab] & \cdots & [af] \\ [ab] & [bb] & \cdots & [bf] \\ \cdot & \cdot & \cdot & \cdot \\ [af] & [bf] & \cdots & [ff] \end{vmatrix}^2}.$$

If the original observations are weighted we must write $[waa]$ for $[aa]$, etc.

Ex. *—Suppose that observations of equal weight give

$$\begin{aligned} x - y + 2z &= 3 \\ 3x + 2y - 5z &= 5 \\ 4x + y + 4z &= 21, \end{aligned}$$

while an observation of weight $\frac{1}{4}$ gives

$$-2x + 6y + 6z = 28.$$

For the last equation we substitute

$$-x + 3y + 3z = 14,$$

and the four equations are now of equal weight.

The normal equations are

$$\left.\begin{aligned} 27x_0 + 6y_0 &= 88 \\ 6x_0 + 15y_0 + z_0 &= 70 \\ y_0 + 54z_0 &= 107 \end{aligned}\right\},$$

which give $\qquad x_0 = \frac{49154}{19899}, \quad y_0 = \frac{2617}{737}, \quad z_0 = \frac{12707}{6633},$

or $\qquad x_0 = 2{\cdot}470, \quad y_0 = 3{\cdot}551, \quad z_0 = 1{\cdot}916.$

The weight of the determination x_0 is w_x, where

$$\frac{1}{w_x} = \frac{A_{11}}{D} = \frac{\begin{vmatrix} 15 & 1 \\ 1 & 54 \end{vmatrix}}{\begin{vmatrix} 27 & 6 & 0 \\ 6 & 15 & 1 \\ 0 & 1 & 54 \end{vmatrix}} = \frac{809}{19899},$$

so $\qquad w_x = \frac{19899}{809},$

and similarly $\qquad w_y = \frac{737}{54}, \quad w_z = \frac{2211}{41}.$

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* This example was used by Gauss himself as an illustration of the Method of Least Squares.

The residuals are

$$-\frac{4960}{19899},\quad -\frac{1320}{19899},\quad \frac{1880}{19899},\quad -\frac{1400}{19899},$$

and therefore the sum of the squares of the residuals is

$$[v^2]=\frac{1600}{19899},$$

so the quadratic mean error to be feared in the determination of x is ϵ_x, where $\epsilon_x^2=\frac{[v^2]}{w_x}=\frac{1294400}{(19899)^2}$, and similarly for ϵ_y^2 and ϵ_z^2.

126. Other Examples of the Method.—The Principle of Least Squares is applied in the first of the following examples to a problem of curve-fitting, and in the second to a geometrical construction.

Ex. 1.—*Find the values of the constants a, b, c, which nearly satisfy the equation*

$$f(x)=a+\frac{b}{x}+\frac{c}{x^2}$$

for values of x between q and p, where $f(x)$ is a given function.

In this case

$$\int_q^p\left\{a+\frac{b}{x}+\frac{c}{x^2}-f(x)\right\}^2 dx$$

is to be made a minimum, so differentiating with respect to a, b, and c, we have the three equations

$$a\int_q^p dx+b\int_q^p\frac{dx}{x}+c\int_q^p\frac{dx}{x^2}=\int_q^p f(x)dx,$$

$$a\int_q^p\frac{dx}{x}+b\int_q^p\frac{dx}{x^2}+c\int_q^p\frac{dx}{x^3}=\int_q^p\frac{f(x)}{x}dx,$$

$$a\int_q^p\frac{dx}{x^2}+b\int_q^p\frac{dx}{x^3}+c\int_q^p\frac{dx}{x^4}=\int_q^p\frac{f(x)}{x^2}dx.$$

From these the values of a, b, c can be determined.

Ex. 2.—*The position of a point in a plane is determined as the intersection of several lines furnished by observation. Owing to errors of observation the lines are not exactly concurrent. To find the most probable position of the point.**

Denote by $d_1, d_2, \ldots, d_n$ the distances of a point of the plane from the n given lines; let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the weights attached to those lines: then the required point is that for which

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* d'Ocagne, *Journ. de l'École Pol.* cah. 63 (1893), p. 1.

$$\lambda_1 d_1^2 + \lambda_2 d_2^2 + \ldots + \lambda_n d_n^2$$

is a minimum. This point may be called the *centre of least squares* of the system.

The centre of least squares is the point for which the function

$$S = \sum_{i=1}^{n} \frac{\lambda_i(a_i x + b_i y + c_i)^2}{a_i^2 + b_i^2}$$

is a minimum: so its co-ordinates are given by

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = 0,$$

or

$$\sum_{i=1}^{n} \frac{\lambda_i a_i(a_i x + b_i y + c_i)}{a_i^2 + b_i^2} = 0, \quad \sum_{i=1}^{n} \frac{\lambda_i b_i(a_i x + b_i y + c_i)}{a_i^2 + b_i^2} = 0. \tag{1}$$

Now if O is any point whatever of the plane, and $O_1', O_2', \ldots, O_n'$ are the images of O with respect to the given lines, the centre of gravity O′ of particles of masses $\lambda_1, \lambda_2, \ldots, \lambda_n$ at $O_1', O_2', \ldots, O_n'$ may be called the *symmetric barycentre* of O with respect to the system of lines.

If O be (X, Y), the symmetric barycentre O′ of O has the co-ordinates (X′, Y′), where

$$X' = X - \frac{2}{L}\sum_{i=1}^{n} \frac{\lambda_i a_i(a_i X + b_i Y + c_i)}{a_i^2 + b_i^2}$$

$$Y' = Y - \frac{2}{L}\sum_{i=1}^{n} \frac{\lambda_i b_i(a_i X + b_i Y + c_i)}{a_i^2 + b_i^2}, \tag{2}$$

where $L = \lambda_1 + \ldots + \lambda_n$.

Now suppose the origin taken at P, the centre of least squares, so that by (1) we have

$$\Sigma \frac{\lambda_i a_i c_i}{a_i^2 + b_i^2} = 0, \quad \Sigma \frac{\lambda_i b_i c_i}{a_i^2 + b_i^2} = 0,$$

and suppose the direction of the axis Px chosen so that

$$\Sigma \frac{\lambda_i a_i b_i}{a_i^2 + b_i^2} = 0.$$

Then the equations (2) become

$$X' = \left(1 - \frac{2}{L}\Sigma \frac{\lambda_i a_i^2}{a_i^2 + b_i^2}\right)X, \quad Y' = \left(1 - \frac{2}{L}\Sigma \frac{\lambda_i b_i^2}{a_i^2 + b_i^2}\right)Y,$$

and since

$$\left(1 - \frac{2}{L}\Sigma \frac{\lambda_i a_i^2}{a_i^2 + b_i^2}\right) + \left(1 - \frac{2}{L}\Sigma \frac{\lambda_i b_i^2}{a_i^2 + b_i^2}\right) = 0,$$

these may be written

$$X' = \mu X, \quad Y' = -\mu Y.$$

Let O″ be the symmetric barycentre of O′.

The co-ordinates of O″ are $X'' = \mu^2 X$, $Y'' = \mu^2 Y$, and therefore *the line* OO″ *passes through* P. Moreover, the lines PO and PO′ are equally inclined to Px, and $\frac{PO'}{PO} = \mu = \frac{PO''}{PO'}$.

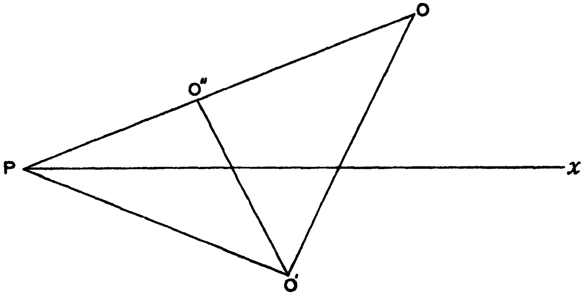

FIG. 17.

Therefore the triangles PO′O″ and POO′ are similar, and O′P makes with O′O″ an angle equal to the angle O′OO″. Thus the centre of least squares P is the intersection of OO″ with the line O′P which makes with O′O″ an angle equal to O′OO″: or otherwise expressed, *the centre of least squares is the intersection of* OO″ *with the tangent at* O′ *to the circle* OO′O″.

127. Case when two Measured Quantities occur in the same Equation of Condition.

—We shall now consider the case in which the quantity n in an equation of condition

$$ax + by + cz + \ldots + ft = n \tag{1}$$

is not itself a measure derived from a single observation to which a known weight is attached, but is a known function of two measures p and q derived from different observations to which weights w_p and w_q are attached. It is required to find the weight which must be attached to the equation (1).

Let the expression for n in terms of p and q be

$$n = \phi(p, q).$$

Then a small error ϵ in p and a small error μ in q give rise to an error $\frac{\partial\phi}{\partial p}\epsilon + \frac{\partial\phi}{\partial q}\mu$ in n; and therefore the weight w_n of n is given by the equation

$$\frac{1}{w_n} = \left(\frac{\partial\phi}{\partial p}\right)^2 \frac{1}{w_p} + \left(\frac{\partial\phi}{\partial q}\right)^2 \frac{1}{w_q}.$$

The equation (1) is now to be treated as if n were a measure derived from a single observation having the weight w_n.

128. **Jacobi's Theorem.**—It was shown by Jacobi[*] that the values of the m unknowns which are obtained from s equations of condition by the Method of Least Squares may be derived also in the following way. Take any m of the equations of condition and solve them as ordinary algebraic equations, obtaining (say) a value

$$x_1 = \frac{A_1}{B_1} \text{(where } A_1 \text{ and } B_1 \text{ are determinants)}$$

for the unknown x. We can select the m equations for this purpose in p ways, where $p = \binom{s}{m}$, and thus obtain p values for x, say

$$x_1 = \frac{A_1}{B_1}, \quad x_2 = \frac{A_2}{B_2}, \quad \ldots, x_p = \frac{A_p}{B_p}.$$

Then *the value of x given by the Method of Least Squares is*

$$x_0 = \frac{A_1B_1 + A_2B_2 + \ldots + A_pB_p}{B_1^2 + B_2^2 + \ldots + B_p^2}.$$

This result is an immediate consequence of Cauchy's well-known theorem on the equivalence of the two forms in which the product of two arrays can be expressed: thus in the case of $s = 3$, $m = 2$, what is to be proved is

$$\frac{\begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1b_1 + a_2b_2 + a_3b_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1^2 + b_2^2 + b_3^2 \end{vmatrix}}{\begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} x_1 b_1 \\ x_2 b_2 \end{vmatrix}\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} + \begin{vmatrix} x_1 b_1 \\ x_3 b_3 \end{vmatrix}\begin{vmatrix} a_1 b_1 \\ a_3 b_3 \end{vmatrix} + \begin{vmatrix} x_2 b_2 \\ x_3 b_3 \end{vmatrix}\begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix}}{\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 b_1 \\ a_3 b_3 \end{vmatrix}^2 + \begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix}^2},$$

which follows at once from the theorem on the product of two arrays.

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[*] *Journ. für Math.* **22** (1841), p. 285.

Ex.[*]—Consider the system of four equations used by Gauss, and worked out in Ex., § 125, above, viz.

$$x - y + 2z = 3, \qquad (1)$$
$$3x + 2y - 5z = 5, \qquad (2)$$
$$4x + y + 4z = 21, \qquad (3)$$
$$-x + 3y + 3z = 14. \qquad (4)$$

Now equations (1), (2), (3) alone would give

$$x = \frac{90}{35}, \quad y = \frac{115}{35}, \quad z = \frac{65}{35},$$

the numerators and denominators being the exact values of the determinants, thus,

$$\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -5 \\ 4 & 1 & 4 \end{vmatrix} = 35, \text{ etc.}$$

Equations (1), (2), (4) would give

$$x = \frac{122}{47}, \quad y = \frac{167}{47}, \quad z = \frac{93}{47}.$$

Equations (1), (3), (4) would give

$$x = \frac{78}{33}, \quad y = \frac{113}{33}, \quad z = \frac{67}{33}.$$

Equations (2), (3), (4) would give

$$x = \frac{-304}{-124}, \quad y = \frac{-444}{-124}, \quad z = \frac{-236}{-124},$$

so
$$x_0 = \frac{(90 \times 35) + (122 \times 47) + (78 \times 33) + (304 \times 124)}{35^2 + 47^2 + 33^2 + 124^2},$$

$$= \frac{49154}{19899}, \text{ as before.}$$

129. Case when the Unknowns are connected by Rigorous Equations.—It frequently happens that the unknowns $x, y, z, \ldots, t$ are not independent, but are connected by rigorous equations: for instance, if x, y, z represent the three angles of a triangle, they are connected by the rigorous equation $x + y + z = \pi$. In order to discuss this case we shall suppose that the unknowns are given as before by a set of linear equations derived from observation

$$\left.\begin{array}{l} a_1x + b_1y + \ldots + f_1t = n_1 \\ a_2x + b_2y + \ldots + f_2t = n_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_sx + b_sy + \ldots + f_st = n_s \end{array}\right\}, \qquad (1)$$

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[*] Glaisher, *Month. Not.* **40** (1880), p. 600.

(where $n_1, \ldots, n_s$ are the observed quantities), together with p rigorous equations

$$\left.\begin{aligned} \phi_1(x, y, \ldots, t) &= 0 \\ \phi_2(x, y, \ldots, t) &= 0 \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \\ \phi_p(x, y, \ldots, t) &= 0 \end{aligned}\right\}. \qquad (2)$$

We shall suppose that (by multiplying each of the equations of condition by the square root of its weight if necessary) the equations of condition have been rendered all of equal weight. Then, as in the proof of § 125, we see that the most probable values of the unknowns are those which make

$$E \equiv [aa]x^2 + [bb]y^2 + \ldots + 2[ab]xy + \ldots - 2[an]x - \ldots + [nn]$$

a minimum, subject to the conditions (2). We must have therefore

$$\frac{\partial E}{\partial x}dx + \frac{\partial E}{\partial y}dy + \ldots + \frac{\partial E}{\partial t}dt = 0,$$

where $dx, dy, \ldots, dt$ are subject only to the conditions

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial x}dx + \frac{\partial \phi_1}{\partial y}dy + \ldots + \frac{\partial \phi_1}{\partial t}dt &= 0 \\ \frac{\partial \phi_2}{\partial x}dx + \frac{\partial \phi_2}{\partial y}dy + \ldots + \frac{\partial \phi_2}{\partial t}dt &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \end{aligned}\right\},$$

and therefore the unknowns are to be determined from the m equations

$$\left.\begin{aligned} \frac{\partial E}{\partial x} + \lambda_1\frac{\partial \phi_1}{\partial x} + \lambda_2\frac{\partial \phi_2}{\partial x} + \ldots + \lambda_p\frac{\partial \phi_p}{\partial x} &= 0 \\ \frac{\partial E}{\partial y} + \lambda_1\frac{\partial \phi_1}{\partial y} + \lambda_2\frac{\partial \phi_2}{\partial y} + \ldots + \lambda_p\frac{\partial \phi_p}{\partial y} &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \\ \frac{\partial E}{\partial t} + \lambda_1\frac{\partial \phi_1}{\partial t} + \lambda_2\frac{\partial \phi_2}{\partial t} + \ldots + \lambda_p\frac{\partial \phi_p}{\partial t} &= 0 \end{aligned}\right\}$$

(where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are unknown multipliers), together with equations (2). We have therefore $(m+p)$ equations to determine the $(m+p)$ unknown quantities $x, y, \ldots, t, \lambda_1, \ldots, \lambda_p$.

Ex. — *The measures of the four angles of a plane quadrangle are $\alpha, \beta, \gamma, \delta$, with weights g_1, g_2, g_3, g_4 respectively. Find the most probable* *values of the angles, and show that the weight of the value found for the first angle is*

$$\frac{g_1g_2g_3+g_1g_2g_4+g_1g_3g_4+g_2g_3g_4}{g_3g_4+g_2g_4+g_2g_3},$$

so that when all the weights are equal, the weight of the final value of an angle is $\frac{4}{3}$ *the weight of an observation.*

Let the angles be θ_1, θ_2, θ_3, θ_4. The equations of condition, reduced to unit weight, are

$$\sqrt{g_1}\theta_1=\sqrt{g_1}\alpha,\quad \sqrt{g_2}\theta_2=\sqrt{g_2}\beta,\quad \sqrt{g_3}\theta_3=\sqrt{g_3}\gamma,\quad \sqrt{g_4}\theta_4=\sqrt{g_4}\delta,$$

and the rigorous condition is that

$$\theta_1+\theta_2+\theta_3+\theta_4=2\pi.$$

We have therefore to make

$$g_1(\theta_1-\alpha)^2+g_2(\theta_2-\beta)^2+g_3(\theta_3-\gamma)^2+g_4(\theta_4-\delta)^2$$

a minimum, subject to the last condition. We have therefore

$$g_1(\theta_1-\alpha)d\theta_1+g_2(\theta_2-\beta)d\theta_2+g_3(\theta_3-\gamma)d\theta+g_4(\theta_4-\delta)d\theta_4=0,$$

where the differentials are subject to the condition

$$d\theta_1+d\theta_2+d\theta_3+d\theta_4=0,$$

and therefore

$$g_1(\theta_1-\alpha)=g_2(\theta_2-\beta)=g_3(\theta_3-\gamma)=g_4(\theta_4-\delta).$$

We thus obtain

$$\theta_1=\alpha-\frac{g_2g_3g_4(\alpha+\beta+\gamma+\delta-2\pi)}{g_2g_3g_4+g_1g_3g_4+g_1g_2g_4+g_1g_2g_3},$$

or

$$\theta_1=\frac{g_1(g_2g_3+g_3g_4+g_4g_2)}{\Sigma g_2g_3g_4}\alpha-\frac{g_2g_3g_4}{\Sigma g_2g_3g_4}(\beta+\gamma+\delta-2\pi),$$

and similar expressions for the other angles.

Denoting the weight of θ_1 by W_1, we have at once

$$\frac{1}{W_1}=\left\{\frac{g_1(g_2g_3+g_3g_4+g_4g_2)}{\Sigma g_2g_3g_4}\right\}^2\frac{1}{g_1}+\left\{\frac{g_2g_3g_4}{\Sigma g_2g_3g_4}\right\}^2\left(\frac{1}{g_2}+\frac{1}{g_3}+\frac{1}{g_4}\right);$$

$$=\frac{g_2g_3+g_3g_4+g_4g_2}{(\Sigma g_2g_3g_4)^2}\{g_1(g_2g_3+g_3g_4+g_4g_2)+g_2g_3g_4\},$$

$$=\frac{g_2g_3+g_3g_4+g_4g_2}{\Sigma g_2g_3g_4},$$

or

$$W_1=\frac{g_2g_3g_4+g_1g_3g_4+g_1g_2g_4+g_1g_2g_3}{g_2g_3+g_3g_4+g_4g_2},$$

which is the required result.

130. The Solving Processes of Gauss and Seidel.—When the number of unknowns is great, the algebraic methods for the solution of the normal equations, which have been described in §§ 117–120, become exceedingly laborious: under these circumstances, the normal equations may be solved by a method of successive approximation in the following way.

Writing the normal equations

$$\left.\begin{array}{l} a_{11}x + a_{12}y + \ldots + a_{1m}t = c_1 \\ a_{12}x + a_{22}y + \ldots + a_{2m}t = c_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{1n}x + a_{2n}y + \ldots + a_{nm}t = c_n \end{array}\right\},$$

we first assume for $x, y, \ldots, t$ any system of values. Since in the normal equations the diagonal coefficients are sums of squares, whereas the non-diagonal coefficients are sums of products of which in general some are positive and some are negative, it is most often found that the diagonal coefficients are larger than the others, and therefore the corresponding terms are most important: so we therefore generally take as initial values for $x, y, \ldots, t$ the numbers

$$\frac{c_1}{a_{11}}, \frac{c_2}{a_{22}}, \ldots, \frac{c_m}{a_{mm}}$$

respectively.

With these assumed values of $x, y, \ldots, t$, we calculate the quantities

$$N_1 = a_{11}x + a_{12}y + \ldots + a_{1m}t - c_1,$$

and take
$$\Delta x = -\frac{N_1}{a_{11}}.$$

We can now assert that by adding Δx to the assumed value of x we are *improving* it: for if we denote by Q the sum of the squares of the residuals when the assumed values are put for $x, y, \ldots, t$, we have

$$\begin{aligned} Q &= a_{11}x^2 + a_{22}y^2 + \ldots + a_{mm}t^2 + 2a_{12}xy + 2a_{13}xz + \ldots \\ &\qquad + 2a_{1m}xt + \ldots - 2c_1x - \ldots + p, \\ &= (a_{11}x + a_{12}y + \ldots + a_{1m}t - c_1)^2/a_{11} \\ &\qquad + \text{terms depending only on } y, z, \ldots, t. \end{aligned}$$

If in this we replace x by $x+\Delta x$, without changing $y, z, \ldots, t$, the effect is to destroy the term $(a_{11}x+\ldots+a_{1m}t-c_1)^2$ in Q without affecting the other terms: that is, the effect is to *diminish* Q. Now the set of values of $x, y, \ldots, t$ which we wish to obtain are the set which make Q a minimum: and therefore if we say that one set of values of $x, y, \ldots, t$ is an *improvement* in another set when it corresponds to a smaller value of Q, we can say that by adding Δx to x, without changing the values of $y, \ldots, t$, we are obtaining an improved set of values for the unknowns.

Now with the improved set of values of $x, y, \ldots, t$ we calculate the quantity

$$N_2 = a_{21}x + a_{22}y + \ldots + a_{2m}t - c_2,$$

and take

$$\Delta y = -\frac{N_2}{a_{22}}.$$

In the same way we can show that to add Δy to the assumed value of y is an improvement. Proceeding in this way, we improve each of the values $x, y, \ldots, t$ in succession, and then return to x. The process, which is due to Seidel,* may be stopped when the residual N's are sufficiently small. In common with all iterative methods of approximation, it has the advantage that an error of calculation continually corrects itself in the subsequent steps of the process.

Ex.—Let us solve by this process the normal equations of Gauss's original example, namely

$$\begin{cases} 27x+6y = 88, \\ 6x+15y+z=70, \\ y+54z = 107. \end{cases}$$

We take as a first approximation

$$\begin{aligned} x &= \tfrac{88}{27} = 3 \text{ roughly}, \\ y &= \tfrac{70}{15} = 5 \text{ roughly}, \\ z &= \tfrac{107}{54} = 2 \text{ roughly}. \end{aligned}$$

Then

$$N_1 = 27x+6y-88 = 23,$$

and

$$\Delta x = -\frac{N_1}{27} = -\frac{23}{27} = -0{\cdot}85.$$

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* *Münch. Abh.* **11** (1874), Abt. 3, p. 81.

Thus the improved set of values is $x=2\cdot15, y=5, z=2$. These give $N_2=6x+15y+z-70=19\cdot9$, and therefore $\Delta y=-\frac{N_2}{15}=-1\cdot33$, so we now have $x=2\cdot15$, $y=3\cdot67, z=2$, and $N_3=y+54z-107=4\cdot67$. Thus $\Delta z=-\frac{N_3}{54}=-0\cdot086$, giving $x=2\cdot15, y=3\cdot67, z=1\cdot914$, and $N_1=-8$.

Repeating the process, we have

$\Delta x=-\frac{N_1}{27}=\frac{8}{27}=0\cdot296$, giving $x=2\cdot446, y=3\cdot67, z=1\cdot914, N_2=1\cdot640$;

$\Delta y=-\frac{N_2}{15}=-0\cdot1093$, giving $x=2\cdot446$, $y=3\cdot561$, $z=1\cdot914$, $N_3=-0\cdot083$;

$\Delta z=-\frac{N_3}{54}=0\cdot01$, giving $x=2\cdot446$, $y=3\cdot561$, $z=1\cdot915$, $N_1=-0\cdot592$;

$\Delta x=-\frac{N_1}{27}=0\cdot022$, giving $x=2\cdot468$, $y=3\cdot561$, $z=1\cdot915$, $N_2=0\cdot138$;

$\Delta y=-\frac{N_2}{15}=-0\cdot0092$, giving $x=2\cdot468$, $y=3\cdot552$, $z=1\cdot915$.

These differ only by at most 0·002 from the true values.

If Seidel's process be carried out for a set of equations with literal coefficients, such as

$$\begin{cases} ax+hy+gz=l, \\ hx+by+fz=m, \\ gx+fy+cz=n, \end{cases}$$

it is readily seen that the value of x is what would be obtained from the formula

$$x=\frac{1}{abc}\begin{vmatrix} l & m & n \\ h & b & f \\ g & f & c \end{vmatrix}\left(1+\frac{2fgh}{abc}-\frac{f^2}{bc}-\frac{g^2}{ac}-\frac{h^2}{ab}\right)^{-1}$$

by expanding the last factor by the binomial theorem as an infinite series.

A method closely akin to this had been communicated many years before by Gauss to Gerling.* It may be illustrated by the following example: †

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* Cf. the appendix to Gerling's work on the application of the calculus of compensation to practical geometry (1843), p. 386. Another very similar process was described by Jacobi, *Ast. Nach.* No. 523 (1845), p. 297.

† C. A. Schott, *U.S. Coast Survey Rep.* (1855), p. 255.

Let the normal equations be

$$\begin{aligned} 0 &= 2{\cdot}8 + 76x - 30y - 20z - 26u, \\ 0 &= -4{\cdot}1 - 30x + 83y - 25x - 28u, \\ 0 &= -1{\cdot}9 - 20x - 25y + 89z - 44u, \\ 0 &= 3{\cdot}2 - 26x - 28y - 44z + 98u. \end{aligned}$$

To ascertain which of the unknowns will probably be the greatest, we examine the quotients:

$$x = -\frac{2{\cdot}8}{76} = -0{\cdot}03, \quad y = \frac{4{\cdot}1}{83} = +0{\cdot}04, \quad z = \frac{1{\cdot}9}{89} = 0{\cdot}02,$$

$$u = -\frac{3{\cdot}2}{98} = -0{\cdot}03.$$

Accordingly we begin with y and write $y = 0{\cdot}04 + \Delta y$. The equations now become

$$\begin{aligned} 0 &= 1{\cdot}60 + 76x - 30\,\Delta y - 20z - 26u, \\ 0 &= -0{\cdot}78 - 30x + 83\,\Delta y - 25z - 28u, \\ 0 &= -2{\cdot}90 - 20x - 25\,\Delta y + 89z - 44u, \\ 0 &= 2{\cdot}08 - 26x - 28\,\Delta y - 44z + 98u. \end{aligned}$$

This gives for quotients (roughly)

$$x = -\frac{1{\cdot}60}{76} = -0{\cdot}02, \quad \Delta y = \frac{0{\cdot}78}{83} = 0{\cdot}01, \quad z = \frac{2{\cdot}90}{89} = 0{\cdot}03,$$

$$u = -\frac{2{\cdot}08}{98} = -0{\cdot}02.$$

We therefore now substitute $z = 0{\cdot}03 + \Delta z$: and proceeding in this way, the whole solution may be brought to the form:

	$y = 0{\cdot}04$	$z = 0{\cdot}03$	$\Delta y = 0{\cdot}01$	$x = -0{\cdot}01$	$u = -0{\cdot}01$	$\Delta x = -0{\cdot}003$	$\Delta z = -0{\cdot}002$	$\Delta u = 0{\cdot}001$
$0 =$	$1{\cdot}60$	$1{\cdot}00$	$0{\cdot}70$	$-0{\cdot}06$	$0{\cdot}20$	$-0{\cdot}028$	$0{\cdot}012$	$-0{\cdot}014$
	$-0{\cdot}78$	$-1{\cdot}53$	$-0{\cdot}70$	$-0{\cdot}40$	$-0{\cdot}12$	$-0{\cdot}030$	$+0{\cdot}020$	$-0{\cdot}008$
	$-2{\cdot}90$	$-0{\cdot}23$	$-0{\cdot}48$	$-0{\cdot}28$	$+0{\cdot}16$	$+0{\cdot}220$	$+0{\cdot}042$	$-0{\cdot}002$
	$+2{\cdot}08$	$+0{\cdot}76$	$+0{\cdot}48$	$+0{\cdot}74$	$-0{\cdot}24$	$-0{\cdot}162$	$-0{\cdot}074$	$-0{\cdot}024$

The operation is now completed if we are satisfied with two places of decimals, and the first unknown quantity is $x + \Delta x + \Delta^2 x + \ldots$ or $x = -0{\cdot}013$.

Similarly

$$\begin{aligned} y &= 0{\cdot}050, \\ z &= 0{\cdot}028, \\ u &= -0{\cdot}009. \end{aligned}$$

131. Alternatives to the Method of Least Squares.—At different times various methods have been proposed, other than the Method of Least Squares, for dealing with problems which are commonly solved by that method. We shall now notice briefly some of these:

1°. **The Method of Tobias Mayer.**—In the latter half of the eighteenth century the most plausible values of the unknowns were commonly found by a method which had been published by Tobias Mayer in 1748 and 1760.* It consisted in arranging the equations of condition into sets, forming the sum of the equations of each set, and treating the equations so obtained as normal equations. The method is decidedly inferior to the Method of Least Squares.

2°. **The Method of Minimum Approximation.**—Consider a system of s incompatible linear equations in m unknowns ($s > m$):

$$(1)\quad a_i x + b_i y + \ldots + f_i t = n_i \qquad (i = 1, 2, \ldots, s).$$

Replace it by the following:

$$(2)\quad a_i x + b_i y + \ldots + f_i t - n_i = r_i \qquad (i = 1, 2, \ldots, s),$$

where the quantities r_i may be called *residuals.* The name *minimum approximation* of the system (1) was given by Goedseels † to the smallest value which we can assign to the absolute value of the greatest residual of the system (2), in order that this system (2) may be compatible.

The problem of determining the minimum approximation of a system (1) was enunciated and solved by Laplace ‡ in 1799, but his method involved such laborious calculations as to be in general impracticable Six years later, in 1805, Legendre proposed the same problem, in the appendix to his *Nouvelles Méthodes pour la détermination des orbites des comètes*: he found no easy method of solution, and proposed to replace the method by that of Least Squares.

A much better solution was given in 1911 by C. J. de la Vallée-Poussin.§

3°. **Edgeworth's Method.**—Taking the data as usual in the form

$$\left.\begin{array}{l} a_1 x + b_1 y + \ldots + f_1 t = n_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_s x + b_s y + \ldots + f_s t = n_s \end{array}\right\}$$

where $n_1, n_2, \ldots, n_s$ are measures of equal weight, F. Y. Edgeworth in 1887 || proposed to define the most plausible values $x, y, \ldots, t$ in the following way: $x, y, \ldots, t$ *are to be such as to render minimum the sum of the absolute values of the residuals,*

$$|a_1 x + \ldots + f_1 t - n_1| + |a_2 x + \ldots + f_2 t - n_2| + \ldots$$
$$+ |a_s x + b_s y + \ldots + f_s t - n_s|.$$

It may readily be shown that this rule is derivable from the hypothesis that the law of error is of the form

$$y = \tfrac{1}{2} h e^{-hx},$$

where x is taken positively in both directions.

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* *Kosmographische Nachrichten und Sammlungen.*

† P. J. E. Goedseels, *Théorie des erreurs d'observation*: Louvain (1907).

‡ *Mécanique céleste*, Livre III. No. 39.

§ *Annales de la Soc. Sc. de Bruxelles*, **35** (1911), B, p. 1.

|| *Phil. Mag.* **24** (1887), p. 222, and **25** (1888), p. 184.

CHAPTER X

PRACTICAL FOURIER ANALYSIS

132. Introduction.—For the description of phenomena and the solution of problems in Physics, Astronomy, and Meteorology, much use is made of *Fourier series*, that is to say, series of the type

$$\begin{aligned} &a_0 + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta \\ &\quad + b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots \qquad (1) \end{aligned}$$

where $a_0, a_1, b_1, a_2, b_2, \ldots$ are independent of θ.

Consider, for example, the vibration of a violin string. Let a stretched elastic string be fixed at its end-points and take the axis of x along the string, so that the abscissae of the end-points can be taken to be $x=0$ and $x=l$; let y be the displacement (in a direction perpendicular to the string), at time t, of the point of the string whose abscissa is x. Then when the string is set into vibration in such a way that it emits its fundamental note, unmixed with any overtones, its vibration is represented mathematically by the equation

$$y = \mathrm{A} \sin \frac{\pi x}{l} \sin (\lambda t + a),$$

where $2\pi/\lambda$ is the period of the note in question, λ depending on the mass, length, and tension of the string, and where A and a are arbitrary constants. If the string is set into vibration in such a way that it emits its first overtone (the octave of the fundamental note) unmixed with any other sound, the vibration is represented by

$$y = \mathrm{B} \sin \frac{2\pi x}{l} \sin (2\lambda t + \beta),$$

while the second overtone is represented by

$$y = \mathrm{C} \sin \frac{3\pi x}{l} \sin (3\lambda t + \gamma),$$

and so on. When the string is set into vibration in a quite general fashion, so that all the overtones are present in the sound emitted, the displacement at time t is represented by a sum of these terms; thus

$$y = \mathrm{A} \sin \frac{\pi x}{l} \sin (\lambda t + \alpha) + \mathrm{B} \sin \frac{2\pi x}{l} \sin (2\lambda t + \beta) + \mathrm{C} \sin \frac{3\pi x}{l} \sin (3\lambda t + \gamma) + \ldots \quad (2)$$

and therefore the velocity at time t is represented by

$$\frac{dy}{dt} = \lambda \mathrm{A} \sin \frac{\pi x}{l} \cos (\lambda t + \alpha) + 2\lambda \mathrm{B} \sin \frac{2\pi x}{l} \cos (2\lambda t + \beta) + 3\lambda \mathrm{C} \sin \frac{3\pi x}{l} \cos (3\lambda t + \gamma) + \ldots \quad (3)$$

Suppose that at the initial instant, $t = 0$, the displacement and velocity at every point of the string are given; let the displacement be $\phi(x)$ and the velocity be $\psi(x)$. Suppose, moreover, that we are in possession of a method which enables us to express a given function $f(x)$, which vanishes at $x = 0$ and $x = l$, as a series of the form

$$f(x) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots \quad (4)$$

where $b_1, b_2, b_3, \ldots$ do not depend on x, but depend upon the nature of the function $f(x)$. Applying this theorem to the function $\phi(x)$, we should have an equation

$$\phi(x) = p_1 \sin \frac{\pi x}{l} + p_2 \sin \frac{2\pi x}{l} + p_3 \sin \frac{3\pi x}{l} + \ldots \quad (5)$$

where $p_1, p_2, p_3, \ldots$ may be regarded as known, since the function $\phi(x)$ is given. Similarly

$$\psi(x) = q_1 \sin \frac{\pi x}{l} + q_2 \sin \frac{2\pi x}{l} + q_3 \sin \frac{3\pi x}{l} + \ldots \quad (6)$$

where $q_1, q_2, q_3, \ldots$ may be regarded as known.

But putting $t=0$ in (2) and (3) we have

$$\phi(x)=\mathrm{A}\sin\alpha\sin\frac{\pi x}{l}+\mathrm{B}\sin\beta\sin\frac{2\pi x}{l}+\mathrm{C}\sin\gamma\sin\frac{3\pi x}{l}+\dots \quad (7)$$

$$\psi(x)=\lambda\mathrm{A}\cos\alpha\sin\frac{\pi x}{l}+2\lambda\mathrm{B}\cos\beta\sin\frac{2\pi x}{l}$$
$$+3\lambda\mathrm{C}\cos\gamma\sin\frac{3\pi x}{l}+\dots \quad (8)$$

Comparing (5) and (7) we have

$$\mathrm{A}\sin\alpha=p_1,\quad \mathrm{B}\sin\beta=p_2,\quad \mathrm{C}\sin\gamma=p_3\ \dots \quad (9)$$

Comparing (6) and (8), we have

$$\lambda\mathrm{A}\cos\alpha=q_1,\quad 2\lambda\mathrm{B}\cos\beta=q_2,\quad 3\lambda\mathrm{C}\cos\gamma=q_3\ \dots \quad (10)$$

The systems of equations (9) and (10) enable us to find the unknown constants A, α, B, β, C, γ, . . . in terms of p_1, q_1, p_2, q_2, that is to say, in terms of known quantities. Thus equation (4) enables us to analyse the initial data into constituents A, α, B, β, . . . such that the first pair of constituents (A, α) gives rise to the fundamental note of the string, the second pair of constituents (B, β) gives rise to the first overtone, the third pair of constituents gives rise to the second overtone, and so on.

As a second illustration consider the Theory of Tides. The tide-generating potential due to the sun and moon may be expanded as a series of terms of the type

$$\mathrm{A}\sin(\lambda t+\epsilon),$$

where t denotes the time and A, λ, ϵ are independent of t but change from one term of this type to another. Each such constituent term gives rise to an oscillation of the sea, the oscillation having the same period $2\pi/\lambda$ as the term in the tide-generating potential to which it is due; and the height of the tide at any instant at any seaport may therefore be represented as a series of terms of the type

$$\mathrm{A}'\sin(\lambda t+\epsilon'),$$

where the constant λ is characteristic of the particular tide but is the same for all seaports, while the constants A$'$ and ϵ' are characteristic of the particular constituent tide and the particular seaport. By analysing the observed tides at a seaport by means of a theorem similar to (4), we can find these constants A' and ϵ'; we are then in a position to predict the tides at this port for all future time.

A series of the type

$$a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

is generally called a *Fourier series*, the coefficients $a_0, a_1, b_1, \ldots$ being called *Fourier coefficients*: and the representation of a given function by means of a series of this form is called *Fourier analysis.* The term *trigonometric interpolation* is perhaps more appropriate when (as in the present chapter) we are concerned only with finding a series with a *finite* number of terms which takes given values for a given *finite* number of values of the argument x.

133. **Interpolation of a Function by a Sine Series.**—In the last article we have seen the importance, in Applied Mathematics, of a theorem which will enable us to analyse a given function into a sum of trigonometric terms. The problem was solved, at any rate in its simplest form, in 1754–59 by Clairaut* and Lagrange,† who showed how *to construct a sum of n trigonometric terms, such as*

$$u(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_{n-1} \sin (n-1)x, \quad (1)$$

which will take given values for $(n-1)$ *given equally-spaced values of the argument* x; say,

$$u\left(\frac{\pi}{n}\right) = u_1, \quad u\left(\frac{2\pi}{n}\right) = u_2, \quad u\left(\frac{3\pi}{n}\right) = u_3, \quad \ldots, \quad u\left(\frac{(n-1)\pi}{n}\right) = u_{n-1},$$

where $u_1, u_2, \ldots, u_{n-1}$ are given numbers.

To effect this, Lagrange remarked that the sum

$$\sin\frac{p\pi}{n}\sin\frac{q\pi}{n} + \sin\frac{2p\pi}{n}\sin\frac{2q\pi}{n} + \ldots + \sin\frac{(n-1)p\pi}{n}\sin\frac{(n-1)q\pi}{n}$$

(where p and q denote positive integers less than n) has the value $\frac{1}{2}n$ when p is equal to q, and is zero when p is not equal to q. Therefore the function

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* Clairaut, *Hist. de l'Acad.*, Paris, 1754, p. 545.

† Lagrange, *Misc. Taurin.* i. (1759), p. 1: reprinted *Œuvres de Lagrange*, i. p. 39; *Misc. Taurin.* iii. (1762–5), p. 258: reprinted *Œuvres de Lagrange*, i. p. 553.

$$\frac{2}{n}\left(\sin x \sin\frac{p\pi}{n} + \sin 2x \sin\frac{2p\pi}{n} + \ldots + \sin(n-1)x \sin\frac{(n-1)p\pi}{n}\right)u_p$$

has the value u_p when $x=\frac{p\pi}{n}$ and vanishes when $x=\frac{q\pi}{n}$ where q is different from p: whence it follows immediately that *the coefficients in* (1) *must be given by the equation*

$$b_m=\frac{2}{n}\left\{u_1 \sin\frac{m\pi}{n} + u_2 \sin\frac{2m\pi}{n} + \ldots + u_{n-1} \sin\frac{(n-1)m\pi}{n}\right\}.$$

It should be noticed that the expression (1) satisfies the condition that:

1°. It takes the prescribed values $u_1, \ldots, u_{n-1}$ at the given values of the argument.

2°. It is periodic with period 2π.

3°. It is an odd function of x.

Suppose now that $u(x)$ is a function of x which takes prescribed values $u_1, \ldots, u_{n-1}$ at the given values of the argument, but suppose that the function $u(x)$ is not periodic and is not odd; in such a case the expression (1) would have a graph agreeing more or less with the graph of the function $u(x)$ between $x=0$ and $x=\pi$, but the agreement would cease altogether for values of x less than zero or greater than π. It is important to realise that by this method of interpolation we can obtain an expression which agrees very closely indeed with a given function over a certain range of values of the argument, but which, outside that range of values, bears no resemblance whatever to the function.

134. A more general Representation of a Trigonometric Series.—In the last article we obtained for the function $u(x)$ an interpolation formula which is formed of sines only. We shall now obtain a more general formula[*] which involves both sines and cosines, and which, moreover, enables us to make the best use of all the data in our possession, when we have more data than the minimum number required.

Let it be required *to find a sum*

$$\left.\begin{aligned}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_r \cos rx\\ + b_1 \sin x + b_2 \sin 2x + \ldots + b_r \sin rx\end{aligned}\right\} \qquad (1)$$

---

[*] Bessel, *Königsberger Beobachtungen*, 1 Abt. p. iii. (1815).

which furnishes the best possible representation of a function $u(x)$, when we are given that $u(x)$ takes the values $u_0, u_1, \ldots, u_{n-1}$ respectively, when x takes the values $0, \frac{2\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}$ respectively: n being some number greater than $2r$. The problem is to determine the $(2r+1)$ constants $a_0, a_1, b_1, \ldots, a_r, b_r$, so as to make the expression (1) take, as nearly as possible, the n values $u_0, u_1, \ldots, u_{n-1}$, when x takes the values $0, \frac{2\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}$; so the equations of condition are

$$\left\{\begin{aligned}
u_0 &= a_0 + a_1 + \ldots + a_r. \\
u_1 &= a_0 + a_1 \cos\frac{2\pi}{n} + a_2 \cos\frac{2.2\pi}{n} + \ldots + a_r \cos\frac{r.2\pi}{n} \\
&\qquad + b_1 \sin\frac{2\pi}{n} + b_2 \sin\frac{2.2\pi}{n} + \ldots + b_r \sin\frac{r.2\pi}{n}. \\
u_2 &= a_0 + a_1 \cos\frac{2.2\pi}{n} + a_2 \cos\frac{4.2\pi}{n} + \ldots + a_r \cos\frac{2r.2\pi}{n} \\
&\qquad + b_1 \sin\frac{2.2\pi}{n} + b_2 \sin\frac{4.2\pi}{n} + \ldots + b_r \sin\frac{2r.2\pi}{n}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u_{n-1} &= a_0 + a_1 \cos\frac{(n-1)2\pi}{n} + \ldots + a_r \cos\frac{r(n-1)2\pi}{n} \\
&\qquad + b_1 \sin\frac{(n-1)2\pi}{n} + \ldots + b_r \sin\frac{r(n-1)2\pi}{n}.
\end{aligned}\right.$$

The normal equation for a_0 is therefore

$$\begin{aligned}
u_0 + u_1 + u_2 + \ldots + u_{n-1} \\
= na_0 &+ a_1\left\{1 + \cos\frac{2\pi}{n} + \ldots + \cos\frac{(n-1)2\pi}{n}\right\} \\
&+ a_2\left\{1 + \cos\frac{2.2\pi}{n} + \ldots + \cos\frac{2(n-1)2\pi}{n}\right\} \\
&+ \ldots \\
&+ a_r\left\{1 + \cos\frac{r.2\pi}{n} + \ldots + \cos\frac{r(n-1)2\pi}{n}\right\} \\
&+ b_1\left\{\sin\frac{2\pi}{n} + \sin\frac{2.2\pi}{n} + \ldots + \sin\frac{(n-1)2\pi}{n}\right\} \\
&+ \ldots \\
&+ b_r\left\{\sin\frac{r.2\pi}{n} + \ldots + \sin\frac{r(n-1)2\pi}{n}\right\}.
\end{aligned}$$

Since

$$1+\cos\frac{2k\pi}{n}+\cos\frac{4k\pi}{n}+\ldots+\cos\frac{2(n-1)k\pi}{n}=0,$$

when k is an integer greater than zero and less than n, and

$$\sin\frac{2k\pi}{n}+\sin\frac{4k\pi}{n}+\ldots+\sin\frac{2(n-1)k\pi}{n}=0,$$

this becomes

$$u_0+u_1+u_2+\ldots+u_{n-1}=na_0.$$

The normal equation for a_1 is

$$\begin{aligned}
u_0+u_1\cos\frac{2\pi}{n}+u_2\cos\frac{2.2\pi}{n}+\ldots+u_{n-1}\cos\frac{(n-1)2\pi}{n}
&=a_0\left\{1+\cos\frac{2\pi}{n}+\ldots+\cos\frac{(n-1)2\pi}{n}\right\}\\
&+a_1\left\{1+\cos^2\frac{2\pi}{n}+\ldots+\cos^2\frac{(n-1)2\pi}{n}\right\}\\
&+a_2\left\{1+\cos\frac{2\pi}{n}\cos\frac{2.2\pi}{n}+\cos\frac{2.2\pi}{n}\cos\frac{4.2\pi}{n}+\ldots\right.\\
&\qquad\left.+\cos\frac{(n-1)2\pi}{n}\cos\frac{2(n-1)2\pi}{n}\right\}\\
&+\ldots\\
&+b_1\left\{\cos\frac{2\pi}{n}\sin\frac{2\pi}{n}+\ldots+\cos\frac{(n-1)2\pi}{n}\sin\frac{(n-1)2\pi}{n}\right\}\\
&+\ldots.
\end{aligned}$$

Using the trigonometric formulae

$$\left\{\begin{aligned}
&1+\cos^2\frac{2\pi}{n}+\cos^2\frac{4\pi}{n}+\ldots+\cos^2\frac{2(n-1)\pi}{n}=\frac{1}{2}n,\\
&1+\cos\frac{2\pi}{n}\cos\frac{2.2\pi}{n}+\cos\frac{2.2\pi}{n}\cos\frac{4.2\pi}{n}+\ldots\\
&\qquad\qquad+\cos\frac{2(n-1)\pi}{n}\cos\frac{4(n-1)\pi}{n}=0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cos\frac{2\pi}{n}\sin\frac{2\pi}{n}+\ldots+\cos\frac{2(n-1)\pi}{n}\sin\frac{2(n-1)\pi}{n}=0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right.$$

this becomes

$$u_0+u_1\cos\frac{2\pi}{n}+u_2\cos\frac{4\pi}{n}+\ldots+u_{n-1}\cos\frac{2(n-1)\pi}{n}=\frac{1}{2}na_1.$$

The other normal equations may be obtained and reduced in the same way; finally *we obtain the following set of values for $a_0, \ldots, a_r, b_1, \ldots, b_r$:*

$$\left.\begin{aligned}a_0&=\frac{1}{n}\sum_{k=0}^{n-1}u_k\\ a_1&=\frac{2}{n}\sum_{k=0}^{n-1}u_k\cos\frac{2k\pi}{n}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\\ a_r&=\frac{2}{n}\sum_{k=0}^{n-1}u_k\cos\frac{2k\pi r}{n}\\ b_1&=\frac{2}{n}\sum_{k=0}^{n-1}u_k\sin\frac{2k\pi}{n}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\\ b_r&=\frac{2}{n}\sum_{k=0}^{n-1}u_k\sin\frac{2kr\pi}{n}\end{aligned}\right\}. \qquad (1)$$

When $r=\frac{1}{2}n$, the factor of a_r in front of the symbol Σ is $\frac{1}{n}$, not $\frac{2}{n}$.

The connection of this result with that of the preceding article is easily seen; in fact, the formulae of § 133 are merely those of the present article adapted to the particular case when the function u is an *odd* periodic function of its argument, so that $u(x)=-u(2\pi-x)$. For if in the formulae (1) above we write $n=2p$ and suppose that $u_1=-u_{2p-1}$, $u_2=-u_{2p-2}$, etc., with $u_0=0$, $u_p=0$, then the formulae (1) become

$$a_0=0,\ a_1=0,\ \ldots,\ a_r=0,$$

$$b_m=\frac{2}{p}\left\{u_1\sin\frac{m\pi}{p}+u_2\sin\frac{2m\pi}{p}+\ldots+u_{p-1}\sin\frac{(p-1)m\pi}{p}\right\},$$

which agree with the formulae of § 133.

135. **The 12-Ordinate Scheme.**—In most cases in practice the number of given values $u_0, u_1, u_2, \ldots$ is either 12 or 24. We shall first consider the case when 12 values are given.

Let it be required, then, *to obtain an expression*

$$a_0+a_1\cos x+a_2\cos 2x+\ldots+a_5\cos 5x+a_6\cos 6x$$
$$+b_1\sin x+b_2\sin 2x+\ldots+b_5\sin 5x$$

which takes given values $u_0, u_1, \ldots, u_{11}$ respectively when x

takes the values $0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \ldots, \frac{11\pi}{6}$ *respectively.* Formulae (1) of the last article, written in full, are now

$$\left.\begin{aligned}
12a_0 &= u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9 + u_{10} + u_{11} \\
6a_1 &= u_0 + u_1 \cdot \frac{\sqrt{3}}{2} + u_2 \cdot \frac{1}{2} - u_4 \cdot \frac{1}{2} - u_5 \cdot \frac{\sqrt{3}}{2} - u_6 - u_7 \cdot \frac{\sqrt{3}}{2} \\
&\qquad - u_8 \cdot \frac{1}{2} + u_{10} \cdot \frac{1}{2} + u_{11} \cdot \frac{\sqrt{3}}{2} \\
6a_2 &= u_0 + u_1 \cdot \frac{1}{2} - u_2 \cdot \frac{1}{2} - u_3 - u_4 \cdot \frac{1}{2} + u_5 \cdot \frac{1}{2} + u_6 + u_7 \cdot \frac{1}{2} \\
&\qquad - u_8 \cdot \frac{1}{2} - u_9 - u_{10} \cdot \frac{1}{2} + u_{11} \cdot \frac{1}{2} \\
6a_3 &= u_0 - u_2 + u_4 - u_6 + u_8 - u_{10} \\
6a_4 &= u_0 - u_1 \cdot \frac{1}{2} - u_2 \cdot \frac{1}{2} + u_3 - u_4 \cdot \frac{1}{2} - u_5 \cdot \frac{1}{2} + u_6 - u_7 \cdot \frac{1}{2} \\
&\qquad - u_8 \cdot \frac{1}{2} + u_9 - u_{10} \cdot \frac{1}{2} - u_{11} \cdot \frac{1}{2} \\
6a_5 &= u_0 - u_1 \cdot \frac{\sqrt{3}}{2} + u_2 \cdot \frac{1}{2} - u_4 \cdot \frac{1}{2} + u_5 \cdot \frac{\sqrt{3}}{2} - u_6 + u_7 \cdot \frac{\sqrt{3}}{2} \\
&\qquad - u_8 \cdot \frac{1}{2} + u_{10} \cdot \frac{1}{2} - u_{11} \cdot \frac{\sqrt{3}}{2} \\
12a_6 &= u_0 - u_1 + u_2 - u_3 + u_4 - u_5 + u_6 - u_7 + u_8 - u_9 + u_{10} - u_{11} \\
6b_1 &= u_1 \cdot \frac{1}{2} + u_2 \cdot \frac{\sqrt{3}}{2} + u_3 + u_4 \cdot \frac{\sqrt{3}}{2} + u_5 \cdot \frac{1}{2} - u_7 \cdot \frac{1}{2} - u_8 \cdot \frac{\sqrt{3}}{2} \\
&\qquad - u_9 - u_{10} \cdot \frac{\sqrt{3}}{2} - u_{11} \cdot \frac{1}{2} \\
6b_2 &= u_1 \cdot \frac{\sqrt{3}}{2} + u_2 \cdot \frac{\sqrt{3}}{2} - u_4 \cdot \frac{\sqrt{3}}{2} - u_5 \cdot \frac{\sqrt{3}}{2} + u_7 \cdot \frac{\sqrt{3}}{2} \\
&\qquad + u_8 \cdot \frac{\sqrt{3}}{2} - u_{10} \cdot \frac{\sqrt{3}}{2} - u_{11} \cdot \frac{\sqrt{3}}{2} \\
6b_3 &= u_1 - u_3 + u_5 - u_7 + u_9 - u_{11} \\
6b_4 &= u_1 \cdot \frac{\sqrt{3}}{2} - u_2 \cdot \frac{\sqrt{3}}{2} + u_4 \cdot \frac{\sqrt{3}}{2} - u_5 \cdot \frac{\sqrt{3}}{2} + u_7 \cdot \frac{\sqrt{3}}{2} \\
&\qquad - u_8 \cdot \frac{\sqrt{3}}{2} + u_{10} \cdot \frac{\sqrt{3}}{2} - u_{11} \cdot \frac{\sqrt{3}}{2} \\
6b_5 &= u_1 \cdot \frac{1}{2} - u_2 \cdot \frac{\sqrt{3}}{2} + u_3 - u_4 \cdot \frac{\sqrt{3}}{2} + u_5 \cdot \frac{1}{2} - u_7 \cdot \frac{1}{2} + u_8 \cdot \frac{\sqrt{3}}{2} \\
&\qquad - u_9 + u_{10} \cdot \frac{\sqrt{3}}{2} - u_{11} \cdot \frac{1}{2}
\end{aligned}\right\} . \quad (1)$$

If

$$\begin{array}{ll}
u_1 + u_{11} = v_1 & u_1 - u_{11} = w_1 \\
u_2 + u_{10} = v_2 & u_2 - u_{10} = w_2 \\
u_3 + u_9 = v_3 & u_3 - u_9 = w_3 \\
u_4 + u_8 = v_4 & u_4 - u_8 = w_4 \\
u_5 + u_7 = v_5 & u_5 - u_7 = w_5
\end{array}$$

the equations take the simpler form

$$\begin{cases}
12a_0 = u_0 + v_1 + v_2 + v_3 + v_4 + v_5 + u_6 \\
6a_1 = u_0 + v_1 \cdot \frac{\sqrt{3}}{2} + v_2 \cdot \frac{1}{2} - v_4 \cdot \frac{1}{2} - v_5 \cdot \frac{\sqrt{3}}{2} - u_6 \\
6a_2 = u_0 + v_1 \cdot \frac{1}{2} - v_2 \cdot \frac{1}{2} - v_3 - v_4 \cdot \frac{1}{2} + v_5 \cdot \frac{1}{2} + u_6 \\
6a_3 = u_0 - v_2 + v_4 - u_6 \\
6a_4 = u_0 - v_1 \cdot \frac{1}{2} - v_2 \cdot \frac{1}{2} + v_3 - v_4 \cdot \frac{1}{2} - v_5 \cdot \frac{1}{2} + u_6 \\
6a_5 = u_0 - v_1 \cdot \frac{\sqrt{3}}{2} + v_2 \cdot \frac{1}{2} - v_4 \cdot \frac{1}{2} + v_5 \cdot \frac{\sqrt{3}}{2} - u_6 \\
12a_6 = u_0 - v_1 + v_2 - v_3 + v_4 - v_5 + u_6 \\
6b_1 = w_1 \cdot \frac{1}{2} + w_2 \cdot \frac{\sqrt{3}}{2} + w_3 + w_4 \cdot \frac{\sqrt{3}}{2} + w_5 \cdot \frac{1}{2} \\
6b_2 = w_1 \cdot \frac{\sqrt{3}}{2} + w_2 \cdot \frac{\sqrt{3}}{2} - w_4 \cdot \frac{\sqrt{3}}{2} - w_5 \cdot \frac{\sqrt{3}}{2} \\
6b_3 = w_1 - w_3 + w_5 \\
6b_4 = w_1 \cdot \frac{\sqrt{3}}{2} - w_2 \cdot \frac{\sqrt{3}}{2} + w_4 \cdot \frac{\sqrt{3}}{2} - w_5 \cdot \frac{\sqrt{3}}{2} \\
6b_5 = w_1 \cdot \frac{1}{2} - w_2 \cdot \frac{\sqrt{3}}{2} + w_3 - w_4 \cdot \frac{\sqrt{3}}{2} + w_5 \cdot \frac{1}{2}.
\end{cases}$$

If we now write

$$\begin{array}{rl}
u_0 + u_6 = p_0 & u_0 - u_6 = q_0 \\
v_1 + v_5 = p_1 & v_1 - v_5 = q_1 \\
v_2 + v_4 = p_2 & v_2 - v_4 = q_2 \\
v_3 = p_3 & \\
w_1 + w_5 = r_1 & w_1 - w_5 = s_1 \\
w_2 + w_4 = r_2 & w_2 - w_4 = s_2 \\
w_3 = r_3 &
\end{array}$$

the equations take the still simpler form

$$\left\{\begin{aligned}
12a_0 &= p_0+p_1+p_2+p_3\\
6a_1 &= q_0+q_1\cdot\frac{\sqrt{3}}{2}+q_2\cdot\frac{1}{2}\\
6a_2 &= p_0+p_1\cdot\frac{1}{2}-p_2\cdot\frac{1}{2}-p_3\\
6a_3 &= q_0-q_2\\
6a_4 &= p_0-p_1\cdot\frac{1}{2}-p_2\cdot\frac{1}{2}+p_3\\
6a_5 &= q_0-q_1\cdot\frac{\sqrt{3}}{2}+q_2\cdot\frac{1}{2}\\
12a_6 &= p_0-p_1+p_2-p_3\\
6b_1 &= r_1\cdot\frac{1}{2}+r_2\cdot\frac{\sqrt{3}}{2}+r_3\\
6b_2 &= s_1\cdot\frac{\sqrt{3}}{2}+s_2\cdot\frac{\sqrt{3}}{2}\\
6b_3 &= r_1-r_3\\
6b_4 &= s_1\cdot\frac{\sqrt{3}}{2}-s_2\cdot\frac{\sqrt{3}}{2}\\
6b_5 &= r_1\cdot\frac{1}{2}-r_2\cdot\frac{\sqrt{3}}{2}+r_3
\end{aligned}\right.$$

Now write

$$\frac{1}{2}p_1=h_1,\quad \frac{1}{2}p_2=h_2,\quad \frac{1}{2}q_2=l_2,\quad \frac{1}{2}r_1=m_1,$$

$$\frac{\sqrt{3}}{2}q_1=l_1,\quad \frac{\sqrt{3}}{2}r_2=m_2,\quad \frac{\sqrt{3}}{2}s_1=n_1,\quad \frac{\sqrt{3}}{2}s_2=n_2.$$

(Note that $\frac{\sqrt{3}}{2}=0{\cdot}866=1-\frac{1}{10}-\frac{1}{30}$, which enables the multiplication by $\frac{\sqrt{3}}{2}$ to be performed mentally.)

Then the equations may be written

$$\begin{aligned}
12a_0 &= p_0+p_2+p_1+p_3 & \qquad 6b_1 &= m_1+r_3+m_2\\
12a_6 &= p_0+p_2-p_1-p_3 & 6b_5 &= m_1+r_3-m_2\\
6a_1 &= q_0+l_2+l_1 & 6a_3 &= q_0-q_2\\
6a_5 &= q_0+l_2-l_1 & 6b_2 &= n_1+n_2\\
6a_4 &= p_0+p_3-h_1-h_2 & 6b_4 &= n_1-n_2\\
6a_2 &= p_0+h_1-p_3-h_2 & 6b_3 &= r_1-r_3
\end{aligned}$$

These equations give the coefficients $a_0, a_1, \ldots, b_5$ in a convenient form; the computation may conveniently be carried out on a printed sheet of which the arrangement is shown in the accompanying specimen:*

Ex. 2.—*Given*

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
0	0·262	0·524	0·786	1·047	1·309	0	− 1·309	− 1·047	− 0·786	− 0·524	− 0·262

express $u(x)$ *as a Fourier series.*

[Ans. $0{\cdot}977 \sin x - 0{\cdot}453 \sin 2x + 0{\cdot}262 \sin 3x - 0{\cdot}151 \sin 4x + 0{\cdot}070 \sin 5x$.]

136. Approximate Formulae for Rapid Calculations.—In many cases the harmonics above the third (*i.e.* those with the coefficients a_4, a_5, a_6, b_4, b_5) may be neglected, the function being capable of representation with sufficient accuracy by an expression

$$\left.\begin{aligned} u(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \\ + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \end{aligned}\right\}. \qquad (1)$$

It was pointed out by S. P. Thompson † that when this is the case we can calculate the coefficients by simple averaging of the data without any multiplications, in the following way:

First, as we have seen in equation (1) of § 135, the coefficients a_0, a_3, b_3 are given by the equations

$$a_0 = \frac{1}{12}(u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9 + u_{10} + u_{11}), \qquad (2)$$

$$a_3 = \frac{1}{6}(u_0 - u_2 + u_4 - u_6 + u_8 - u_{10}), \qquad (3)$$

$$b_3 = \frac{1}{6}(u_1 - u_3 + u_5 - u_7 + u_9 - u_{11}). \qquad (4)$$

Next, we have, by putting x successively equal to 90° and 270° in equation (1) above,

$$\left.\begin{aligned} u_3 = a_0 - a_2 + b_1 - b_3 \\ u_9 = a_0 - a_2 - b_1 + b_3 \end{aligned}\right\}, \qquad (5)$$

whence $$u_3 - u_9 = 2(b_1 - b_3),$$

---

* The computation form has been designed by help of the suggestions derived from many writers, among whom particular mention should be made of C. Runge, *Zeits. für Math. u. Phys.* **48** (1903), p. 443.

† *Proc. Phys. Soc. London*, **23** (1911), p. 334.

and therefore $$b_1 = \tfrac{1}{2}(u_3 - u_9) + b_3. \tag{6}$$

Next, putting x equal to 0° and 180° in (1), we have

$$\left.\begin{aligned} u_0 &= a_0 + a_1 + a_2 + a_3 \\ u_6 &= a_0 - a_1 + a_2 - a_3 \end{aligned}\right\}, \tag{7}$$

whence $$u_0 - u_6 = 2(a_1 + a_3),$$

and therefore $$a_1 = \tfrac{1}{2}(u_0 - u_6) - a_3. \tag{8}$$

Equations (5) and (7) now give

$$a_2 = \tfrac{1}{4}(u_0 - u_3 + u_6 - u_9). \tag{9}$$

We have still to find b_2. For this we shall suppose that the complete graph of the function $u(x)$ is known, so that we can read off the ordinates at any point on it; let us read off the values at $x = 45$,° 135°, 225°, 315° and call them $\bar{u}_1, \bar{u}_3, \bar{u}_5, \bar{u}_7$ respectively. Then, putting $n = 8$ in equation (1) of § 134, we have

$$b_2 = \tfrac{1}{4} \sum_{k=0}^{7} \bar{u}_k \sin \frac{k\pi}{2}$$

or $$b_2 = \tfrac{1}{4}(\bar{u}_1 - \bar{u}_3 + \bar{u}_5 - \bar{u}_7). \tag{10}$$

Equations (2), (3), (4), (6), (8), (9), (10) *give the coefficients* a_0, a_1, a_2, a_3, b_1, b_2, b_3 *merely by forming averages of the measured ordinates of the graph of* $u(x)$.

Ex.—To find an approximate formula for the Fourier series which represents the following observations:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
2·714	3·042	2·134	1·273	0·788	0·495	0·370	0·540	0·191	− 0·357	− 0·437	0·767

Forming the sum of the entries, we have

$$\begin{aligned} 12a_0 &= u_0 + u_1 + u_2 + \ldots + u_{11} \\ &= 11{\cdot}520, \end{aligned}$$

so $$a_0 = 0{\cdot}960,$$

and we now form the following sums:

$$\begin{aligned} 6a_3 &= u_0 - u_2 + u_4 - u_6 + u_8 - u_{10}, && \text{whence } a_3 = 0{\cdot}271 \\ 6b_3 &= u_1 - u_3 + u_5 - u_7 + u_9 - u_{11}, && \text{whence } b_3 = 0{\cdot}1 \\ b_1 &= \tfrac{1}{2}(u_3 - u_9) + b_3, && \text{whence } b_1 = 0{\cdot}915 \\ a_1 &= \tfrac{1}{2}(u_0 - u_6) - a_3, && \text{whence } a_1 = 0{\cdot}901 \\ 4a_2 &= u_0 - u_3 + u_6 - u_9, && \text{whence } a_2 = 0{\cdot}542. \end{aligned}$$

Finally, forming a graph of the function u_x and reading off the ordinates $\bar{u}_1$, $\bar{u}_3$, $\bar{u}_5$, $\bar{u}_7$, corresponding to the arguments $x = 45°$, 135°, 225°, 315° respectively, we have

$$\begin{aligned} 4b_2 &= \bar{u}_1 - \bar{u}_3 + \bar{u}_5 - \bar{u}_7 \\ &= 2{\cdot}36, \end{aligned}$$

whence $$b_2 = 0{\cdot}59 \text{ (approx.)}.$$

Thus an approximate formula for the required Fourier series is

$$0{\cdot}960 + 0{\cdot}90 \cos x + 0{\cdot}54 \cos 2x + 0{\cdot}27 \cos 3x$$
$$+ 0{\cdot}92 \sin x + 0{\cdot}59 \sin 2x + 0{\cdot}1 \sin 3x.$$

137. **The 24-Ordinate Scheme.**—We shall next consider the case when 24 values of $u(x)$ are given, corresponding to $x=0$, $15°$, $30°$, . . ., $345°$. Denoting these by u_0, u_1, . . ., u_{23}, the problem is *to obtain an expression*

$$a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_{12} \cos 12x$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_{11} \sin 11x$$

which takes given values u_0, u_1, . . ., u_{23} *respectively when* x *takes the values* $0, \frac{\pi}{12}, \ldots, \frac{23\pi}{12}$ *respectively.*

If we form the sums and differences of the u's, thus:

$$\begin{array}{lccccccc}
 & u_0 & u_1 & u_2 & u_3 & \ldots & u_{11} & u_{12} \\
 & & u_{23} & u_{22} & u_{21} & \ldots & u_{13} & \\
\hline
\text{Sums} & v_0 & v_1 & v_2 & v_3 & \ldots & v_{11} & v_{12} \\
\text{Differences} & & w_1 & w_2 & w_3 & \ldots & w_{11} &
\end{array}$$

then the formulae (1) of § 134 applied to this case may be written

$$24a_0 = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + v_{10} + v_{11} + v_{12}$$
$$12a_1 = v_0 + v_1 \cos 15° + v_2 \cos 30° + v_3 \cos 45° + v_4 \cos 60°$$
$$+ v_5 \cos 75° + v_6 \cos 90° + v_7 \cos 105° + v_8 \cos 120°$$
$$+ v_9 \cos 135° + v_{10} \cos 150° + v_{11} \cos 165° + v_{12} \cos 180°$$

. . . .

$$12b_1 = w_1 \sin 15° + w_2 \sin 30° + w_3 \sin 45° + w_4 \sin 60° + w_5 \sin 75°$$
$$+ w_6 \sin 90° + w_7 \sin 105° + w_8 \sin 120° + w_9 \sin 135°$$
$$+ w_{10} \sin 150° + w_{11} \sin 165°$$

. . . .

Now form the sums and differences of the v's thus:

$$\begin{array}{ccccccc}
v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & \\
\hline
p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\
q_0 & q_1 & q_2 & q_3 & q_4 & q_5 &
\end{array},$$

and form the sum and differences of the *w*'s thus:

$$
\begin{array}{cccccc}
w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\
w_{11} & w_{10} & w_9 & w_8 & w_7 & \\
\hline
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\
s_1 & s_2 & s_3 & s_4 & s_5 &
\end{array}
$$

Then the equations become

$$24a_0 = p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6$$

$$\begin{aligned}12a_1 = q_0 + q_1 \cos 15^\circ + q_2 \cos 30^\circ + q_3 \cos 45^\circ + q_4 \cos 60^\circ \\ + q_5 \cos 75^\circ\end{aligned}$$

$$\begin{aligned}12a_2 = p_0 + p_1 \cos 30^\circ + p_2 \cos 60^\circ + p_3 \cos 90^\circ + p_4 \cos 120^\circ \\ + p_5 \cos 150^\circ + p_6 \cos 180^\circ\end{aligned}$$

$$\begin{aligned}12a_3 = q_0 + q_1 \cos 45^\circ + q_2 \cos 90^\circ + q_3 \cos 135^\circ + q_4 \cos 180^\circ \\ + q_5 \cos 225^\circ\end{aligned}$$

$$\begin{aligned}12a_4 = p_0 + p_1 \cos 60^\circ + p_2 \cos 120^\circ + p_3 \cos 180^\circ + p_4 \cos 240^\circ \\ + p_5 \cos 300^\circ + p_6 \cos 360^\circ\end{aligned}$$

$$\begin{aligned}12a_5 = q_0 + q_1 \cos 75^\circ + q_2 \cos 150^\circ + q_3 \cos 225^\circ + q_4 \cos 300^\circ \\ + q_5 \cos 375^\circ\end{aligned}$$

$$\begin{aligned}12a_6 = p_0 + p_1 \cos 90^\circ + p_2 \cos 180^\circ + p_3 \cos 270^\circ + p_4 \cos 360^\circ \\ + p_5 \cos 450^\circ + p_6 \cos 540^\circ\end{aligned}$$

$$\begin{aligned}12a_7 = q_0 + q_1 \cos 105^\circ + q_2 \cos 210^\circ + q_3 \cos 315^\circ + q_4 \cos 420^\circ \\ + q_5 \cos 525^\circ\end{aligned}$$

$$\begin{aligned}12a_8 = p_0 + p_1 \cos 120^\circ + p_2 \cos 240^\circ + p_3 \cos 360^\circ + p_4 \cos 480^\circ \\ + p_5 \cos 600^\circ + p_6 \cos 720^\circ\end{aligned}$$

$$\begin{aligned}12a_9 = q_0 + q_1 \cos 135^\circ + q_2 \cos 270^\circ + q_3 \cos 405^\circ + q_4 \cos 540^\circ \\ + q_5 \cos 675^\circ\end{aligned}$$

$$\begin{aligned}12a_{10} = p_0 + p_1 \cos 150^\circ + p_2 \cos 300^\circ + p_3 \cos 450^\circ + p_4 \cos 600^\circ \\ + p_5 \cos 750^\circ + p_6 \cos 900^\circ\end{aligned}$$

$$\begin{aligned}12a_{11} = q_0 + q_1 \cos 165^\circ + q_2 \cos 330^\circ + q_3 \cos 495^\circ + q_4 \cos 660^\circ \\ + q_5 \cos 825^\circ\end{aligned}$$

$$\begin{aligned}24a_{12} = p_0 + p_1 \cos 180^\circ + p_2 \cos 360^\circ + p_3 \cos 540^\circ + p_4 \cos 720^\circ \\ + p_5 \cos 900^\circ + p_6 \cos 1080^\circ\end{aligned}$$

$$\begin{aligned}12b_1 = r_1 \sin 15^\circ + r_2 \sin 30^\circ + r_3 \sin 45^\circ + r_4 \sin 60^\circ + r_5 \sin 75^\circ \\ + r_6 \sin 90^\circ\end{aligned}$$

$$12b_2 = s_1 \sin 30^\circ + s_2 \sin 60^\circ + s_3 \sin 90^\circ + s_4 \sin 120^\circ + s_5 \sin 150^\circ$$

$$\begin{aligned}12b_3 = r_1 \sin 45^\circ + r_2 \sin 90^\circ + r_3 \sin 135^\circ + r_4 \sin 180^\circ \\ + r_5 \sin 225^\circ + r_6 \sin 270^\circ\end{aligned}$$

$$12b_4 = s_1 \sin 60^\circ + s_2 \sin 120^\circ + s_4 \sin 240^\circ + s_5 \sin 300^\circ$$

$$12b_5 = r_1 \sin 75^\circ + r_2 \sin 150^\circ + r_3 \sin 225^\circ + r_4 \sin 300^\circ + r_5 \sin 375^\circ + r_6 \sin 450^\circ$$

$$12b_6 = s_1 \sin 90^\circ + s_3 \sin 270^\circ + s_5 \sin 450^\circ$$

$$12b_7 = r_1 \sin 105^\circ + r_2 \sin 210^\circ + r_3 \sin 315^\circ + r_4 \sin 420^\circ + r_5 \sin 525^\circ + r_6 \sin 630^\circ$$

$$12b_8 = s_1 \sin 120^\circ + s_2 \sin 240^\circ + s_4 \sin 480^\circ + s_5 \sin 600^\circ$$

$$12b_9 = r_1 \sin 135^\circ + r_2 \sin 270^\circ + r_3 \sin 405^\circ + r_4 \sin 540^\circ + r_5 \sin 675^\circ + r_6 \sin 810^\circ$$

$$12b_{10} = s_1 \sin 150^\circ + s_2 \sin 300^\circ + s_3 \sin 450^\circ + s_4 \sin 600^\circ + s_5 \sin 750^\circ$$

$$12b_{11} = r_1 \sin 165^\circ + r_2 \sin 330^\circ + r_3 \sin 495^\circ + r_4 \sin 660^\circ + r_5 \sin 825^\circ + r_6 \sin 990^\circ$$

Now form the sums and differences of the p's, thus:

$$\begin{array}{rcccc} & p_0 & p_1 & p_2 & p_3 \\ & p_6 & p_5 & p_4 & \\ \hline \text{Sums} & l_0 & l_1 & l_2 & l_3 \\ \text{Differences} & m_0 & m_1 & m_2 & \end{array}$$

Then we have

$$\begin{aligned} 24a_0 &= l_0 + l_1 + l_2 + l_3, \\ 12a_2 &= m_0 + m_1 \cos 30^\circ + m_2 \cos 60^\circ, \\ 12a_4 &= l_0 + l_1 \cos 60^\circ + l_2 \cos 120^\circ + l_3 \cos 180^\circ, \\ 12a_6 &= m_0 + m_1 \cos 90^\circ + m_2 \cos 180^\circ, \\ 12a_8 &= l_0 + l_1 \cos 120^\circ + l_2 \cos 240^\circ + l_3 \cos 360^\circ, \\ 12a_{10} &= m_0 + m_1 \cos 150^\circ + m_2 \cos 300^\circ, \\ 24a_{12} &= l_0 + l_1 \cos 180^\circ + l_2 \cos 360^\circ + l_3 \cos 540^\circ, \end{aligned}$$

or

$$\begin{aligned} 24a_0 &= l_0 + l_1 + l_2 + l_3, \\ 12a_4 &= l_0 + \frac{1}{2}l_1 - \frac{1}{2}l_2 - l_3, \\ 12a_8 &= l_0 - \frac{1}{2}l_1 - \frac{1}{2}l_2 + l_3, \\ 24a_{12} &= l_0 - l_1 + l_2 - l_3, \\ 12a_2 &= m_0 + m_1\frac{\sqrt{3}}{2} + m_2\frac{1}{2}, \\ 12a_6 &= m_0 - m_2, \\ 12a_{10} &= m_0 - m_1\frac{\sqrt{3}}{2} + m_2\frac{1}{2}. \end{aligned}$$

Next form the sums and differences of the s's, thus:

$$\begin{array}{r|ccc}
 & s_1 & s_2 & s_3 \\
 & s_5 & s_4 & \\
\hline
\text{Sums} & k_1 & k_2 & k_3 \\
\text{Differences} & n_1 & n_2 &
\end{array}.$$

Then we have

$$\left\{\begin{aligned}
12b_2 &= k_1 \sin 30^\circ + k_2 \sin 60^\circ + k_3 \sin 90^\circ, \\
12b_4 &= n_1 \sin 60^\circ + n_2 \sin 120^\circ, \\
12b_6 &= k_1 \sin 90^\circ + k_3 \sin 270^\circ, \\
12b_8 &= n_1 \sin 120^\circ + n_2 \sin 240^\circ, \\
12b_{10} &= k_1 \sin 150^\circ + k_2 \sin 300^\circ + k_3 \sin 450^\circ,
\end{aligned}\right.$$

or

$$\left\{\begin{aligned}
12b_2 &= k_1 \cdot \frac{1}{2} + k_2 \cdot \frac{\sqrt{3}}{2} + k_3, \\
12b_6 &= k_1 - k_3, \\
12b_{10} &= k_1 \cdot \frac{1}{2} - k_2 \cdot \frac{\sqrt{3}}{2} + k_3, \\
12b_4 &= n_1 \cdot \frac{\sqrt{3}}{2} + n_2 \cdot \frac{\sqrt{3}}{2}, \\
12b_8 &= n_1 \cdot \frac{\sqrt{3}}{2} - n_2 \cdot \frac{\sqrt{3}}{2}.
\end{aligned}\right.$$

Now write

$$\frac{1}{2}l_1 = l_1', \quad \frac{1}{2}l_2 = l_2', \quad \frac{1}{2}m_2 = m_2', \quad \frac{1}{2}k_1 = k_1',$$

$$\frac{1}{2}\sqrt{3}m_1 = m_1', \quad \frac{1}{2}\sqrt{3}k_2 = k_2', \quad \frac{1}{2}\sqrt{3}n_1 = n_1', \quad \frac{1}{2}\sqrt{3}n_2 = n_2'.$$

Then the equations become

$$\begin{aligned}
24a_0 &= (l_0 + l_2) + (l_1 + l_3), \\
24a_{12} &= (l_0 + l_2) - (l_1 + l_3), \\
12a_4 &= (l_0 + l_1') - (l_2' + l_3), \\
12a_8 &= (l_0 + l_3) - (l_1' + l_2'), \\
12a_2 &= (m_0 + m_2') + m_1', \\
12a_{10} &= (m_0 + m_2') - m_1', \\
12a_6 &= m_0 - m_2, \\
12b_2 &= (k_1' + k_3) + k_2', \\
12b_{10} &= (k_1' + k_3) - k_2', \\
12b_6 &= k_1 - k_3, \\
12b_4 &= n_1' + n_2', \\
12b_8 &= n_1' - n_2'.
\end{aligned}$$

In order to calculate the coefficients with odd suffixes we write 45° – 30° for 15°, etc., and so obtain

$$12a_1 = \left(q_0 + \frac{q_3}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{q_1}{\sqrt{2}} + q_4 - \frac{q_5}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{q_1}{\sqrt{2}} + q_2 + \frac{q_5}{\sqrt{2}}\right),$$

$$12a_3 = \left(q_0 + \frac{q_1}{\sqrt{2}}\right) - \frac{q_3}{\sqrt{2}} - q_4 - \frac{q_5}{\sqrt{2}},$$

$$12a_5 = q_0 - \frac{q_3}{\sqrt{2}} + \frac{1}{2}\left(-\frac{q_1}{\sqrt{2}} + q_4 + \frac{q_5}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{q_1}{\sqrt{2}} - q_2 + \frac{q_5}{\sqrt{2}}\right),$$

$$12a_7 = q_0 + \frac{q_3}{\sqrt{2}} + \frac{1}{2}\left(\frac{q_1}{\sqrt{2}} + q_4 - \frac{q_5}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(-\frac{q_1}{\sqrt{2}} - q_2 - \frac{q_5}{\sqrt{2}}\right),$$

$$12a_9 = q_0 - \frac{q_1}{\sqrt{2}} + \frac{q_3}{\sqrt{2}} - q_4 + \frac{q_5}{\sqrt{2}},$$

$$12a_{11} = q_0 - \frac{q_3}{\sqrt{2}} + \frac{1}{2}\left(-\frac{q_1}{\sqrt{2}} + q_4 + \frac{q_5}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(-\frac{q_1}{\sqrt{2}} + q_2 - \frac{q_5}{\sqrt{2}}\right).$$

Now write

$$\frac{q_1}{\sqrt{2}} = q_1', \quad \frac{q_3}{\sqrt{2}} = q_3', \quad \frac{q_5}{\sqrt{2}} = q_5',$$

$$q_1' + q_5' = t_1, \quad \frac{1}{2}q_4 = q_4', \quad \frac{\sqrt{3}}{2}q_2 = q_2',$$

$$q_1' - q_5' = t_2, \quad \frac{1}{2}t_2 = t_2', \quad \frac{\sqrt{3}}{2}t_1 = t_1',$$

$$t_1' + t_2' = f_1, \quad q_4' + q_2' = e_1,$$

$$t_1' - t_2' = f_2, \quad q_4' - q_2' = e_2.$$

Then the equations become

$$\begin{cases} 12a_1 = q_0 + e_1 + q_3' + f_1, \\ 12a_{11} = q_0 + e_1 - q_3' - f_1, \\ 12a_3 = q_0 - q_4 + t_2 - q_3', \\ 12a_9 = q_0 - q_4 - t_2 + q_3', \\ 12a_5 = q_0 + e_2 - q_3' + f_2, \\ 12a_7 = q_0 + e_2 + q_3' - f_2. \end{cases}$$

We have also

$$12b_1 = \left(r_6 + \frac{r_3}{\sqrt{2}}\right) + \frac{1}{2}\left(-\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} + r_2\right) + \frac{\sqrt{3}}{2}\left(\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} + r_4\right),$$

$$12b_3 = \frac{r_1}{\sqrt{2}} + r_2 + \frac{r_3}{\sqrt{2}} - \frac{r_5}{\sqrt{2}} - r_6,$$

$$12b_5 = r_6 - \frac{r_3}{\sqrt{2}} + \frac{1}{2}\left(\frac{r_1}{\sqrt{2}} - \frac{r_5}{\sqrt{2}} + r_2\right) + \frac{\sqrt{3}}{2}\left(\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} - r_4\right),$$

$$12b_7 = -r_6 - \frac{r_3}{\sqrt{2}} + \frac{1}{2}\left(\frac{r_1}{\sqrt{2}} - \frac{r_5}{\sqrt{2}} - r_2\right) + \frac{\sqrt{3}}{2}\left(\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} + r_4\right),$$

$$12b_9 = \frac{r_1}{\sqrt{2}} - r_2 + \frac{r_3}{\sqrt{2}} - \frac{r_5}{\sqrt{2}} + r_6,$$

$$12b_{11} = -r_6 + \frac{r_3}{\sqrt{2}} + \frac{1}{2}\left(-\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} - r_2\right) + \frac{\sqrt{3}}{2}\left(\frac{r_1}{\sqrt{2}} + \frac{r_5}{\sqrt{2}} - r_4\right).$$

Put $\frac{r_1}{\sqrt{2}} = r_1', \quad \frac{r_3}{\sqrt{2}} = r_3', \quad \frac{r_5}{\sqrt{2}} = r_5',$

$$r_1' + r_5' = h_1, \quad \frac{1}{2}h_2 = h_2', \quad \frac{\sqrt{3}}{2}h_1 = h_1', \quad r_3' - h_2' = j_1,$$

$$r_1' - r_5' = h_2, \quad \frac{1}{2}r_2 = r_2', \quad \frac{\sqrt{3}}{2}r_4 = r_4', \quad r_6 + r_2' = j_2.$$

Then the above equations become

$$\left\{\begin{aligned} 12b_1 &= (j_1 + h_1') + (j_2 + r_4'), \\ 12b_{11} &= (j_1 + h_1') - (j_2 + r_4'), \\ 12b_3 &= (h_2 + r_3') + (r_2 - r_6), \\ 12b_9 &= (h_2 + r_3') - (r_2 - r_6), \\ 12b_5 &= (h_1' - j_1) + (j_2 - r_4'), \\ 12b_7 &= (h_1' - j_1) - (j_2 - r_4'). \end{aligned}\right.$$

The calculation is made on a computing form arranged as shown in the sheets inset:

Ex. 2.—*Find the Fourier expression for* $u(\theta)$, *given*

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}
300	401	373	241	73	−56	−100	−66	0	41	27	−34

u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}
−100	−127	−100	−41	0	−18	−100	−207	−273	−241	−100	107

[Answer, 100 ($\sin\theta + \cos\theta + \sin 2\theta + \cos 2\theta + \sin 3\theta + \cos 3\theta$).]

138. Application of the Method to Observational Data.—1°. *Adjustment to Period.*—In applying the above-described method of practical Fourier analysis to observational data, we have first to find the values of the argument at which the data $u_0, u_1, u_2, \ldots, u_{23}$ are to be taken. Suppose, for instance, that the observed period of the phenomenon is 185·28 days. Then, since $\frac{1}{24}$ of 185·28 is 7·72, if we take u_0 to be the value of the

observed quantity at the instant t_0, we must take u_1 to be its value at the instant $t_0 + 7\cdot72$ days, while u_2 will be its value at the instant $t_0 + 15\cdot44$ days, u_3 will be its value at the instant $t_0 + 23\cdot16$ days, and so on.

2°. *Allowance for Secular Change.*—In many cases the phenomenon whose variation is to be studied is not strictly periodic: thus if the numbers to be analysed represent hourly means of some meteorological phenomenon, the means for hour 0 will not in general be the same as the means for hour 24. This difference is allowed for in practice by applying a correction to each of the terms except that for noon.

3°. *Allowance for the use of Means.*—In many cases the data from which the Fourier expansion is to be computed are not the actual values of the ordinates corresponding to the values $0, \frac{2\pi}{n} \ldots$ of the argument, but the *mean values* of the ordinates taken over certain *intervals.* Thus if we wish to find the curve which represents the annual variation of temperature at a given station, we generally take as data the mean temperatures of the twelve separate months. It is evident, however, that if we were to calculate the curve directly from these data, taking the temperature on the middle day of July to be the mean temperature of July, we should introduce an error, since the average temperature on the middle day of July is not the same as the average temperature over the whole month: in fact, the curve obtained from the means would be too low in summer and too high in winter, the true curve being external to the curve of means.

We can deal with this difficulty by applying a correction to the data in the following way:

Let m_{p-1}, m_p, m_{p+1} be three successive means, each taken over an interval 2ϵ; and let u_p be the true value of the function for the middle of the interval over which m_p is taken, so that u_p is the quantity which should be substituted for m_p as a datum from which to construct the Fourier representation.

We shall suppose that the function may be represented with sufficient accuracy for values of the argument in this region by an expression

$$u = a + 2bx + 3cx^2,$$

where a, b, c are constants, and where x is the argument measured from the middle of the interval over which m_p is measured: we have therefore

$$u_p = a, \quad m_{p-1} = \frac{1}{2\epsilon}\int_{-3\epsilon}^{-\epsilon} u\,dx, \quad m_p = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} u\,dx, \quad m_{p+1} = \frac{1}{2\epsilon}\int_{\epsilon}^{3\epsilon} u\,dx.$$

Performing the integrations, these equations give

$$\begin{aligned} m_{p-1} &= a - 4b\epsilon + 13c\epsilon^2, \\ m_{p+1} &= a + 4b\epsilon + 13c\epsilon^2, \\ m_p &= a + c\epsilon^2, \end{aligned}$$

whence $$c\epsilon^2 = -\frac{1}{12}\left(m_p - \frac{m_{p+1} + m_{p-1}}{2}\right) = \frac{1}{24}\delta^2 m_p,$$

and therefore

$$u_p = m_p - \frac{1}{24}\delta^2 m_p = m_p + \frac{1}{12}\left(m_p - \frac{m_{p+1} + m_{p-1}}{2}\right).$$

That is to say, *in order to convert the given mean m_p into the true ordinate corresponding to the middle of the interval over which m_p is taken, we add a correction equal to one-twelfth of the excess of m_p over the mean of m_{p+1} and m_{p-1}.*

An alternative method is to compute the Fourier expression from the given means and then multiply all its periodic terms (*i.e.* all its terms except the constant a_0) by a factor which represents the ratio of the amplitude of the true curve to the amplitude of the curve of means.

An Example of Harmonic Analysis.—In the accompanying example the data are taken from observations of the magnitude of the variable star RW Cassiopeiae;* the magnitude of the variable star is denoted by m.

The curve represented by the harmonic formula is drawn in the figure. The observed magnitudes of RW Cassiopeiae are also given in Fig. 18 for purposes of comparison.

139. Probable Error of the Fourier Coefficients.—Knowing the probable errors of observation affecting the data $u_0, u_1, u_2, \ldots$, we can easily calculate the probable errors of the values deduced for the Fourier coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$. For (§§ 89, 94) if ϵ is a linear function of $u_0, u_1, \ldots, u_n$, say $\lambda_0 u_0 + \lambda_1 u_1 + \ldots + \lambda_n u_n$, and if the probable error of each of the quantities $u_0, u_1, \ldots, u_n$ is q, then the probable error of ϵ is $(\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2)^{\frac{1}{2}} q$. Thus in the 24-ordinate scheme, since $24a_0 = u_0 + u_1 + u_2 + \ldots + u_{23}$, the probable error of a_0 is $q/\sqrt{24}$ or $0{\cdot}204q$.

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* E. T. Whittaker and C. Martin, *Monthly Notices*, *R.A.S.* 71 (1911), p. 511.

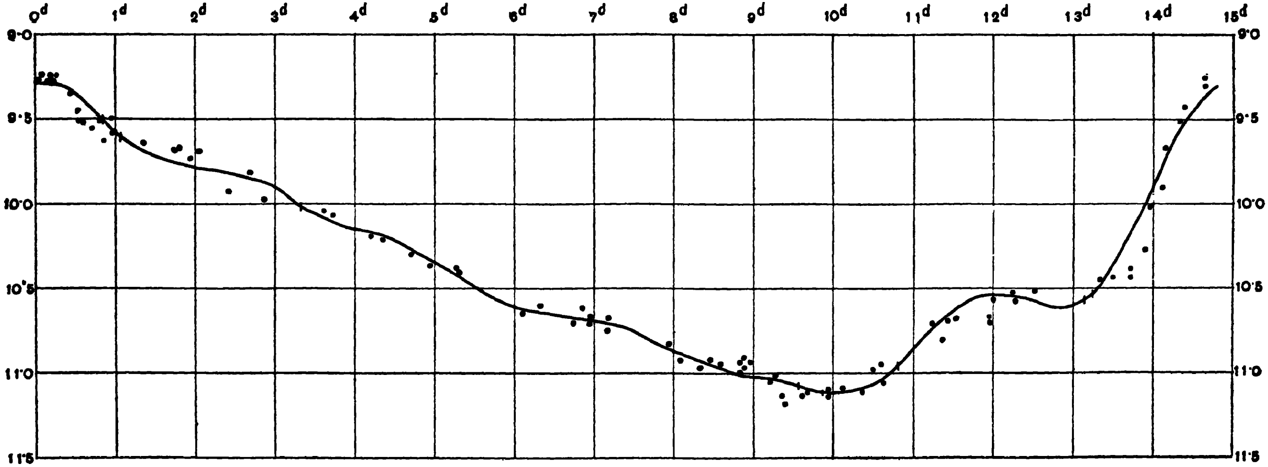

FIG. 18.—Observed magnitudes of RW Cassiopeiae, compared with the curve whose equation is

$$m = 10{\cdot}36 + 0{\cdot}70 \sin(\theta + 233°) + 0{\cdot}17 \sin(2\theta + 260°) + 0{\cdot}08 \sin(3\theta + 234°) + 0{\cdot}13 \sin(4\theta + 249°) + 0{\cdot}09 \sin(5\theta + 280°) + 0{\cdot}06 \sin(6\theta + 304°).$$

Period used, 14·81 days. Observations falling exactly on the curve are denoted by strokes.

A sine term and a cosine term in the Fourier representation, which have the same period, are often combined into a single term thus,

$$a \cos \theta + b \sin \theta = \mathrm{R} \sin (\theta + \alpha),$$

where $\mathrm{R} = \surd(a^2 + b^2)$ and $\cos \alpha = b/\mathrm{R}$, $\sin \alpha = a/\mathrm{R}$: we then require to know the probable error of R and α, assuming that the probable errors of a and b have already been calculated. Suppose that the probable errors of a and b are each of amount ϵ. Then, since

$$\delta \mathrm{R} = \frac{a\delta a + b\delta b}{\surd(a^2 + b^2)},$$

we see that *the probable error in* R *is* ϵ. Moreover, since

$$\delta \alpha = \frac{b\delta a - a\delta b}{a^2 + b^2},$$

the probable error in α is

$$\frac{\epsilon}{\surd(a^2 + b^2)}.$$

It is important to have clear ideas on this subject, since otherwise there is a risk of carrying the computations to more digits than is warranted by the degree of accuracy of the data. This remark applies particularly to the computation of α.

140. Trigonometric Interpolation for Unequal Intervals of the Argument.—Lastly, we shall consider the representation of a function $u(x)$ by a trigonometric interpolation formula, when the values of the function are known only for a set of values $a, b, c, \ldots, m, n$ of the argument, which are not at equal intervals apart.

The problem, which is analogous to Lagrange's problem in ordinary interpolation (§ 17), may be solved in more than one way: it is readily seen, indeed, that any of the following expressions will serve:

1°.

$$u(x) = \frac{\sin \frac{1}{2}(x-b) \sin \frac{1}{2}(x-c) \ldots \sin \frac{1}{2}(x-n)}{\sin \frac{1}{2}(a-b) \sin \frac{1}{2}(a-c) \ldots \sin \frac{1}{2}(a-n)} u(a) + \ldots$$
$$+ \frac{\sin \frac{1}{2}(x-a) \sin \frac{1}{2}(x-b) \ldots \sin \frac{1}{2}(x-m)}{\sin \frac{1}{2}(n-a) \sin \frac{1}{2}(n-b) \ldots \sin \frac{1}{2}(n-m)} u(n).^*$$

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[*] Cauchy, *Comptes rendus*, **12** (1841), p. 283 = *Œuvres* (1), **6**, p. 71.

2°.

$$u(x)=\frac{(\cos x-\cos b)(\cos x-\cos c)\ .\ .\ .\ (\cos x-\cos n)}{(\cos a-\cos b)(\cos a-\cos c)\ .\ .\ .\ (\cos a-\cos n)}u(a)+.\ .\ .$$
$$+\frac{(\cos x-\cos a)(\cos x-\cos b)\ .\ .\ .\ (\cos x-\cos m)}{(\cos n-\cos a)(\cos n-\cos b)\ .\ .\ .\ (\cos n-\cos m)}u(n).$$

3°.

$$u(x)=\frac{\sin x(\cos x-\cos b)(\cos x-\cos c)\ldots(\cos x-\cos n)}{\sin a(\cos a-\cos b)(\cos a-\cos c)\ldots(\cos a-\cos n)}u(a)+\ldots$$
$$+\frac{\sin x(\cos x-\cos a)\ .\ .\ .\ (\cos x-\cos m)}{\sin n(\cos n-\cos a)\ .\ .\ .\ (\cos n-\cos m)}u(n).*$$

4°.

$$u(x)=\frac{\sin(x-b)\sin(x-c)\ .\ .\ .\ \sin(x-n)}{\sin(a-b)\sin(a-c)\ .\ .\ .\ \sin(a-n)}u(a)+.\ .\ .$$
$$+\frac{\sin(x-a)\sin(x-b)\ .\ .\ .\ \sin(x-m)}{\sin(n-a)\sin(n-b)\ .\ .\ .\ \sin(n-m)}u(n).\dagger$$

Ex. 1.—Discuss the relation of the above formulae to the formulae obtained for equal intervals of the argument in §§ 133, 134.

Ex. 2.—By making b tend to equality with a in formula 1° above, obtain an interpolation formula for the case when the function and its first derivative are both known at $x=a$, while the function alone is known at $x=c, d, \ldots$.

(This is useful when we happen to know the values of the argument for which the function is a maximum or a minimum.)

Ex. 3.—Apply the first of the above formulae to obtain an expression for

$$\int_0^{2\pi} u(x)dx.$$

(Use the formula

$$\int_0^{2\pi}\sin\frac{1}{2}(x-x_1)\sin\frac{1}{2}(x-x_2)\ldots\sin\frac{1}{2}(x-x_{2p})\,dx=\frac{\pi}{2^{2p-1}}\Sigma\cos\left(\frac{1}{2}s-s_p\right),$$

where s denotes $x_1+x_2+\ .\ .\ .+x_{2p}$, and s_p is the sum of p of these x's.)

[Baillaud, *Toulouse Ann.* ii. (1886), B.]

MISCELLANEOUS EXAMPLES ON CHAPTER X

In each of the following examples it is required to find a Fourier series for $u(x)$. The tabulated values of $u(x)$ represent equidistant ordinates spaced at intervals of $\frac{1}{24}$ of the complete period.

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* These two formulae are due to Gauss, Nachlass, *Werke* (1866), iii. pp. 291, 292.

† Hermite, *Cours d'analyse.*

Ex. 1.—

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}
4	4	38	68	79	92	91	75	94	118	115	73	85	86	78

u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}
77	86	74	43	50	35	35	20	− 10

Ex. 2.—

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
1	11	50	105	138	158	159	150	139	142	143	127	134

u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}
140	135	150	160	134	120	120	100	72	66	30

Ex. 3.—

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}
25	4	35	60	45	50	45	68	118	148	150	122	125	139

u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}
125	85	82	60	24	20	28	45	56	40

Ex. 4.—

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
170	142	160	171	129	139	130	150	181	179	170	148	106

u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}
75	74	7	15	34	48	69	105	130	146	160

ADDITIONAL REFERENCE

G. C. Danielson and C. Lanczos, " Some improvements in practical Fourier analysis ", *J. Franklin Inst.* **233** (1942), pp. 365, 435.

[*Copies of the Computation Sheets facing pages 270 and 278 may be obtained direct from the Publishers in quantities of not less than one dozen, at the price of 2s. 6d. per dozen sets. Special terms will be quoted for orders of one thousand and upwards.*]

CHAPTER XI

GRADUATION, OR THE SMOOTHING OF DATA

141. The Problem of Graduation.—Suppose that as a result of observation or experience of some kind we have obtained a set of values of a variable u corresponding to equidistant values of its argument: let these values be denoted by $u_1, u_2, u_3, \ldots, u_n$. If they have been derived from observations of some natural phenomenon, they will be affected by errors of observation; if they are statistical data derived from the examination of a comparatively small field, they will be affected by irregularities arising from the accidental peculiarities of the field; that is to say, if we examine another field and derive a set of values of u from it, the set of values of u derived from the two fields will not in general agree with each other. In any case, if we form a table of the differences $\Delta u_1 = u_2 - u_1$, $\Delta u_2 = u_3 - u_2, \ldots$, $\Delta^2 u_1 = \Delta u_2 - \Delta u_1$, etc., it will generally be found that these differences are irregular, so that the difference table cannot be used for the purposes to which a difference table is usually put, viz. finding interpolated values of u, or differential coefficients of u with respect to its argument, or definite integrals involving u. Before we can use the difference table, we must perform a process of "smoothing"; that is to say, we must find another sequence $u_1', u_2', \ldots, u_n'$ whose terms differ as little as possible from the terms of the sequence $u_1, u_2, \ldots, u_n$, but which has regular differences. This smoothing process, leading to the formation of $u_1', u_2', \ldots, u_n'$, is called the *graduation* or *adjustment* of the observations.

For example, let us consider an extract from the Government Female Annuitants (1883) Ultimate Table and form a difference table of the entries. We have, denoting by q_x the probability of a person aged x dying in a year,

x (Age).	$10^5 q_x$.	Δ.
50	1019	
		531
51	1550	
		61
52	1611	
		142
53	1753	
		19
54	1772	
		−224
55	1548	
		474
56	2022	
		−99
57	1923	
		−81
58	1842	
		487
59	2329	

The differences are altogether irregular, and, as we shall see (§ 155), when the data are adjusted the differences become more regular.

In dealing with experimental results of no great accuracy the smoothing process is generally performed graphically,* but in the present chapter we shall be concerned with more refined methods involving analytical formulae.

142. Woolhouse's Formula of Graduation.—We shall consider first a formula which, though now disused in practice, is of considerable historical and theoretical interest. Woolhouse proposed † to pass five ordinary parabolas through the five sets of points

$$(u_{-7}, u_{-2}, u_3), (u_{-6}, u_{-1}, u_4), (u_{-5}, u_0, u_5), (u_{-4}, u_1, u_6), (u_{-3}, u_2, u_7),$$

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* For the best graphical method cf. T. B. Sprague, *J.I.A.* **26** (1886), p. 77. For valuable comments on such methods cf. Whewell, *Novum Organum Renovatum*, Book III. ch. vii. p. 204 of the edition of 1858.

† *J.I.A.* **15** (1870), p. 389. In *J.I.A.* **23** (1882), p. 351, G. F. Hardy showed that the calculations required for the application of Woolhouse's formula might be performed by a "columnar" process of calculation, in the form

$$u_x' = \tfrac{1}{125}\Sigma(1-3\delta^2)\Delta_5{}^3\Sigma^2 u,$$

which is identical with the form $(\frac{1}{125})[5]^3(1-3\delta^2)u$, given later by him, since $\Delta_5\Sigma = [5]$; and T. G. Ackland, *J.I.A.* **23** (1882), p. 352, showed that they can be performed by summations of a different nature.

and take as the graduated value of u the arithmetic mean of the values derived from these five parabolas. Now using Δ_5 to denote the operation Δ performed with the interval 5, by the ordinary interpolation formula (§ 21), the values derived from these five parabolas are:

$$u_0 = u_{-2} + \frac{2}{5}\Delta_5 u_{-2} - \frac{3}{25}\Delta_5{}^2 u_{-7} \qquad \text{(from } u_{-7},\ u_{-2},\ u_3\text{)},$$

$$u_0 = u_{-1} + \frac{1}{5}\Delta_5 u_{-1} - \frac{2}{25}\Delta_5{}^2 u_{-6} \qquad \text{(from } u_{-6},\ u_{-1},\ u_4\text{)},$$

$$u_0 = u_0 \qquad \text{(from } u_{-5},\ u_0,\ u_5\text{)},$$

$$u_0 = u_1 - \frac{1}{5}\Delta_5 u_1 + \frac{3}{25}\Delta_5{}^2 u_{-4} \qquad \text{(from } u_{-4},\ u_1,\ u_6\text{)},$$

$$u_0 = u_2 - \frac{2}{5}\Delta_5 u_2 + \frac{7}{25}\Delta_5{}^2 u_{-3} \qquad \text{(from } u_{-3},\ u_2,\ u_7\text{)}.$$

Therefore if $u_0{}'$ denote the graduated value of u_0, we have

$$5u_0{}' = [5]u_0 + \frac{2}{5}\Delta_5 u_{-2} + \frac{1}{5}\Delta_5 u_{-1} - \frac{1}{5}\Delta_5 u_1 - \frac{2}{5}\Delta_5 u_2 - \frac{3}{25}\Delta_5{}^2 u_{-7}$$
$$- \frac{2}{25}\Delta_5{}^2 u_{-6} + \frac{3}{25}\Delta_5{}^2 u_{-4} + \frac{7}{25}\Delta_5{}^2 u_{-3}, \qquad (1)$$

where $[5]u_0$ stands for $u_{-2} + u_{-1} + u_0 + u_1 + u_2$.

The operation of replacing a term u_r by the sum

$$u_{r-\frac{n-1}{2}} + \ldots + u_r + \ldots + u_{r+\frac{n-1}{2}}$$

will be called *summation by n's*, and denoted by the operator $[n]$, so that

$$[n]u_0 = u_{-\frac{n-1}{2}} + \ldots + u_0 + \ldots + u_{\frac{n-1}{2}}.$$

We shall use $[n]^2$ to denote the effect of performing this operation twice in succession, so that, for example,

$$[5]^2 u_0 = u_4 + 2u_3 + 3u_2 + 4u_1 + 5u_0 + 4u_{-1} + 3u_{-2} + 2u_{-3} + u_{-4}.$$

Since $\Delta_5 u_{-2} = [5]\Delta u_0$, and $\Delta_5{}^2 u_{-4} = [5]^2\Delta^2 u_0$, equation (1) may be written

$$\frac{5}{[5]}u_0{}' = u_0 + \frac{2}{5}\Delta u_0 + \frac{1}{5}\Delta u_1 - \frac{1}{5}\Delta u_3 - \frac{2}{5}\Delta u_4 - \frac{3}{25}[5]\Delta^2 u_{-3} - \frac{2}{25}[5]\Delta^2 u_{-2}$$
$$+ \frac{3}{25}[5]\Delta^2 u_0 + \frac{7}{25}[5]\Delta^2 u_1$$

$$= -\frac{2}{5}[5]u_3 + \frac{3}{5}[5]u_2 - \frac{3}{25}[5]\Delta^2 u_{-3} - \frac{2}{25}[5]\Delta^2 u_{-2} + \frac{3}{25}[5]\Delta^2 u_0$$
$$+ \frac{7}{25}[5]\Delta^2 u_1,$$

so
$$\frac{125}{[5]^2}u_0' = -10u_3 + 15u_2 - 3\Delta^2 u_{-3} - 2\Delta^2 u_{-2} + 3\Delta^2 u_0 + 7\Delta^2 u_1$$
$$= -3u_{-3} + 4u_{-2} + u_{-1} + u_0 + u_1 + 4u_2 - 3u_3$$
$$= -3[5]u_{-1} + 7[5]u_0 - 3[5]u_1.$$

Thus Woolhouse's formula of graduation may be written in the forms

$$u_x' = \frac{[5]^3}{125}(-3u_{x-1} + 7u_x - 3u_{x+1}),$$

or * $$u_x' = \frac{[5]^3}{125}(u_x - 3\delta^2 u_x), \quad \text{or } u_x' = \frac{[5]^3}{125}\{10[1] - 3[3]\}u_x,$$

or $$u_x' = \frac{1}{125}\{25u_x + 24(u_{x-1} + u_{x+1}) + 21(u_{x-2} + u_{x+2})$$
$$+ 7(u_{x-3} + u_{x+3}) + 3(u_{x-4} + u_{x+4}) - 2(u_{x-6} + u_{x+6})$$
$$- 3(u_{x-7} + u_{x+7})\},$$

or $$u_x' = 0{\cdot}200u_x + 0{\cdot}192(u_{x-1} + u_{x+1}) + 0{\cdot}168(u_{x-2} + u_{x+2})$$
$$+ 0{\cdot}056(u_{x-3} + u_{x+3}) + 0{\cdot}024(u_{x-4} + u_{x+4})$$
$$- 0{\cdot}016(u_{x-6} + u_{x+6}) - 0{\cdot}024(u_{x-7} + u_{x+7}).$$

143. Summation Formulae.†—The formulae of Woolhouse may be regarded as a particular instance of a class of graduation formulae, much used by actuaries, which may be called *summation formulae*, and which are based on the following principle.

Let Δ denote the operation of differencing, so that $\Delta u_x = u_{x+1} - u_x$; and, as in § 142, let $[2m+1]u_x$ denote the sum of $(2m+1)$ u's of which u_x is the middle one. Then it is possible to find combinations of these operations Δ and [ ] which, when differences above a certain order are neglected, merely reproduce the functions operated on; so that we have (say)

$$f\{\Delta, [\ \ ]\}u_x = u_x + \text{high differences.}$$

We now take $f\{\Delta, [\ \ ]\}u_x$ to be the graduated value of u_x, that is,

$$u_x' = f\{\Delta, [\ \ ]\}u_x,$$

the merit of this u_x' depending on the circumstance that

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* This form is due to Hardy, *J.I.A.* **32** (1896), p. 372.

† On Summation Formulae cf. G. J. Lidstone, *J.I.A.* **41** (1907), p. 348, **42** (1908), p. 106.

$f\{\Delta, [\ \ ]\}u_x$ involves a large number of the observed u's, whose errors to a considerable extent neutralise each other and so produce a smoothed value u_x' in place of u_x.

In practice, instead of the symbol Δ, it is generally convenient to use the symbol of central differencing δ, where $\delta^2 u_0$ denotes $u_1 - 2u_0 + u_{-1}$. Writing $E = e^{2i\phi}$, we have

$$\delta^2 = E - 2 + E^{-1} = -4\sin^2\phi,$$

so that

$$\begin{aligned} &u_{-m} + u_{-m+1} + \ldots + u_0 + \ldots + u_m \\ &= \{e^{-2mi\phi} + e^{-2(m-1)i\phi} + \ldots + e^{-2i\phi} + 1 + e^{2i\phi} + \ldots + e^{2mi\phi}\}u_0 \\ &= (1 + 2\cos 2\phi + 2\cos 4\phi + \ldots + 2\cos 2m\phi)u_0 \\ &= \frac{\sin(2m+1)\phi}{\sin\phi}u_0, \end{aligned}$$

and therefore

$$\begin{aligned} [n]u_0 &= \frac{\sin n\phi}{\sin\phi}u_0 \\ &= \left\{n - \frac{n(n^2-1^2)}{3!}\sin^2\phi + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^4\phi - \ldots\right\}u_0 \end{aligned}$$

or $$\frac{[n]}{n}u_0 = u_0 + \frac{n^2-1^2}{2^2 . 3!}\delta^2 u_0 + \frac{(n^2-1^2)(n^2-3^2)}{2^4 . 5!}\delta^4 u_0 + \ldots$$

This shows that $\frac{[p][q][r]}{p . q . r}u_0 = u_0 + \frac{\Sigma(p^2-1)}{24}\delta^2 u_0 + \text{terms in } \delta^4 u_0$, $\delta^6 u_0, \ldots$ and therefore *a summation formula, correct to third differences,* is

$$u_0' = \frac{[p][q][r]}{p . q . r}\left\{1 - \frac{\Sigma(p^2-1)}{24}\delta^2\right\}u_0.$$

Taking any two formulae of this type, and eliminating δ^2, we obtain summation formulae of a type first introduced (but otherwise demonstrated) by J. A. Higham, *J.I.A.* **23** (1882), p. 335; **24** (1883), p. 44; **25** (1884–85), pp. 15, 245.

Formulae correct to fourth differences may be deduced by the above method. The use of a formula correct to too low an order may lead to systematic distortion of the results.

It may be remarked that formulae such as Woolhouse's, which are based on interpolations, may all be reduced to the summation type; but the converse is not true, so the summation method is the more general.

144. Spencer's Formula.—Perhaps the best of the summation formulae of graduation correct to third differences is the 21-term formula of Spencer,* namely,

$$u_0' = \frac{[5][5][7]}{5\,.\,5\,.\,7}(1 - 4\delta^2)u_0.$$

This may evidently be obtained by taking $p=5$, $q=5$, $r=7$ in the preceding formula. We shall now obtain its expanded expression. If we perform summation by 7's on $2(1 - 4\delta^2 - 3\delta^4 - \frac{1}{2}\delta^6)$ or $(-E^3 + E + 2 + E^{-1} - E^{-3})$, we obtain

$$-E^6 - E^5 + 2E^3 + 3E^2 + 3E + 2 + 3E^{-1} + 3E^{-2} + 2E^{-3} - E^{-5} - E^{-6},$$

and if on this we perform summation by 5's twice, we obtain

$$\begin{aligned}&-E^{10} - 3E^9 - 5E^8 - 5E^7 - 2E^6 + 6E^5 + 18E^4 + 33E^3 + 47E^2 + 57E\\ &\quad + 60 + 57E^{-1} + 47E^{-2} + 33E^{-3} + 18E^{-4} + 6E^{-5} - 2E^{-6} - 5E^{-7}\\ &\quad - 5E^{-8} - 3E^{-9} - E^{-10}.\end{aligned}$$

Spencer's formula may therefore be written

$$\begin{aligned}u_0' = \tfrac{1}{350}\{&60u_0 + 57(u_{-1} + u_1) + 47(u_{-2} + u_2) + 33(u_{-3} + u_3)\\ &+ 18(u_{-4} + u_4) + 6(u_{-5} + u_5) - 2(u_{-6} + u_6) - 5(u_{-7} + u_7)\\ &- 5(u_{-8} + u_8) - 3(u_{-9} + u_9) - (u_{-10} + u_{10})\}\end{aligned}$$

or

$$\begin{aligned}u_0' = {}&0{\cdot}171u_0 + 0{\cdot}163(u_1 + u_{-1}) + 0{\cdot}134(u_2 + u_{-2}) + 0{\cdot}094(u_3 + u_{-3})\\ &+ 0{\cdot}051(u_4 + u_{-4}) + 0{\cdot}017(u_5 + u_{-5}) - 0{\cdot}006(u_6 + u_{-6})\\ &- 0{\cdot}014(u_7 + u_{-7}) - 0{\cdot}014(u_8 + u_{-8}) - 0{\cdot}009(u_9 + u_{-9})\\ &- 0{\cdot}003(u_{10} + u_{-10}).\end{aligned}$$

In the practical application of the formula we form the expression

$$\tfrac{1}{2}(-u_3 + u_1 + 2u_0 + u_{-1} - u_{-3}),$$

sum by 7's and divide by 7, then sum twice by 5's, dividing by 5 each time.

The following is an example of the working process of Spencer's 21-term formula: †

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* This was employed in the graduation of the rates of mortality exhibited by the Manchester Unity Experience, 1893–97. Cf. *J.I.A.* **38** (1904), p. 334; **41** (1907), p. 361.

† J. Spencer, *J.I.A.* **38** (1904), p. 339.

Age (x).	(1) Ungraduated q_x.	(2) Divide by 7 $=u$.	(3) Sum in 3's.	(4) (2)+(3)	(5) $u_{-3}+u_3$.	(6) (4)-(5)	(7) Sum Col. 6 in 7's.	(8) Divide by 5.	(9) Sum Col. 8 in 5's.	(10) Graduated q_x=sum in 5's cutting down as far as necessary.
20	0·00431	62								
1	0·00409	58	181	239						
2	0·00429	61	179	240						
3	0·00422	60	197	257	128	129				
4	0·00530	76	208	284	129	155				
25	0·00505	72	214	286	136	150				
6	0·00459	66	209	275	140	135	997	199		
7	0·00499	71	212	283	160	123	1034	207		
8	0·00526	75	226	301	157	144	1043	209	1055	
9	0·00563	80	239	319	158	161	1069	214	1101	
30	0·00587	84	249	333	167	166	1130	226	1154	·00582
1	0·00595	85	261	346	182	164	1223	245	1220	·00614
2	0·00647	92	273	365	189	176	1302	260	1294	·00648
3	0·00669	96	295	391	195	196	1375	275	1370	·00682
4	0·00746	107	312	419	203	216	1439	288	1438	·00716
35	0·00760	109	327	436	213	223	1508	302	1501	·00749
6	0·00778	111	338	449	215	234	1566	313	1559	
7	0·00828	118	350	468	238	230	1616	323	1620	
8	0·00846	121	358	479	246	233	1667	333		
9	0·00836	119	371	490	256	234	1745	349		
40	0·00916	131	387	518	272	246				
1	0·00956	137	413	550	283	267				
2	0·01014	145	436	581	280	301				
3	0.01076	154	461	615						
4	0·01134	162	477	639						
45	0·01124	161								

145. **Graduation Formulae obtained by fitting a Polynomial.**—We shall next consider a class of graduation formulae which are based on wholly different principles from the summation formulae. Supposing the ungraduated values u_x to be plotted as points against the corresponding values of x, we shall fit a parabolic curve of some assigned degree j to the points $(u_{-n}, u_{-n+1}, \ldots, u_0, \ldots, u_n)$, determining the constants of the curve by the Method of Least Squares, and we shall then take the ordinate of this curve at $x=0$ as the graduated value of u_0.*

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* Cf. Sheppard, *Proc. V. Int. Cong.* (1912), (ii) 348; *Proc. L.M.S.*(2) **13** 97; *J.I.A.* **48** 181, 390, **49** 148; Sherriff, *Proc. R.S.E.* (1920) 112; Condon, *Calif. Publ.* **2** (1927) 55; Birge and Shea, *ibid.*, 67.

Let us then find *the polynomial of degree* j,

$$u(x) = c_0 + c_1x + c_2x^2 + \ldots + c_jx^j,$$

for which $\sum\limits_{p=-n}^{n} \left\{u_p - u(p)\right\}^2$ *is a minimum; then* $u(0)$ *will be taken to be the graduated value of* u_0.

The equations of condition to determine c_0, c_1, . . ., c_j are

$$\begin{cases} c_0 + c_1n + c_2n^2 + \ldots + c_jn^j & = u_n \\ c_0 + c_1(n-1) + c_2(n-1)^2 + \ldots + c_j(n-1)^j & = u_{n-1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ c_0 - c_1n + c_2n^2 - \ldots + c_j(-n)^j & = u_{-n}. \end{cases}$$

If we now form the normal equations, it is evident that the coefficient of c_0 in every alternate equation vanishes, the sums of the odd powers of the natural numbers from $-m$ to $+m$ being zero. Let $j = 2k$ or $2k + 1$. Denote by Σ a summation over the values from $-n$ to n inclusive; let Σs^p be denoted by Σ_p* and let Σs^pu_s, which is the pth *moment*, be denoted by M_p. Then the normal equations which involve c_0 are

$$\begin{cases} c_0\Sigma_0 + c_2\Sigma_2 + \ldots + c_{2k}\Sigma_{2k} & = M_0, \\ c_0\Sigma_2 + c_2\Sigma_4 + \ldots + c_{2k}\Sigma_{2k+2} & = M_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ c_0\Sigma_{2k} + c_2\Sigma_{2k+2} + \ldots + c_{2k}\Sigma_{4k} & = M_{2k} \end{cases}$$

Solving these equations for c_0, we have

$$u_0' = u(0) = c_0 = \frac{\begin{vmatrix} M_0 & M_2 & \ldots & M_{2k} \\ \Sigma_2 & \Sigma_4 & \ldots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_6 & \ldots & \Sigma_{2k+4} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma_{2k} & \Sigma_{2k+2} & \ldots & \Sigma_{4k} \end{vmatrix}}{\begin{vmatrix} \Sigma_0 & \Sigma_2 & \ldots & \Sigma_{2k} \\ \Sigma_2 & \Sigma_4 & \ldots & \Sigma_{2k+2} \\ \Sigma_4 & \Sigma_6 & \ldots & \Sigma_{2k+4} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma_{2k} & \Sigma_{2k+2} & \ldots & \Sigma_{4k} \end{vmatrix}}.$$

---

* From § 69, we note that the sum of the pth powers of the natural numbers from $-n$ to n inclusive (where p is an even number) is given by

$$\Sigma_p = 2\{1^p + 2^p + \ldots + n^p\}$$
$$= 2\left[\frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{p}{2!}B_1n^{p-1} - \frac{p(p-1)(p-2)}{4!}B_2n^{p-3} + \frac{p(p-1)(p-2)(p-3)(p-4)}{6!}B_3n^{p-5} - \ldots\right]$$

where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, . . .

This is the Graduated Value of u_0.—Since the moments M_0, M_2, . . . are linear functions of the ungraduated data u_{-n}, . . ., u_n, we see that the graduated value u_0 is also a linear function of these data.

The following table is useful in performing the computations.

TABLE OF THE SUMS OF POWERS OF THE NATURAL NUMBERS

$$\Sigma_p = (-m)^p + (-m+1)^p + \ldots + (m-1)^p + m^p.$$

$\iota =$	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	
Σ_2	2	10	28	60	110	182	280	408	570	770	Σ_2
Σ_4	2	34	196	708	1958	4550	9352	17544	30666	50666	Σ_4
Σ_6	2	130	1588	9780	41030	134342	369640	893928	1956810	3956810	Σ_6
Σ_8	2	514	13636	144708	925958	4285190	15814792	49369224	135462666	335462666	Σ_8
Σ_{10}	2	2050	120148	2217300	21748550	142680902	707631400	2855115048	9828683850	29828683850	Σ_{10}
Σ_{12}	2	8194	1071076	34625508	522906758	4876471430	32559045832	169997999304	734857072266	2734857072266	Σ_{12}
Σ_{14}	2	32770	9598708	546469620	12753500870	169481829062	1525927974760	10322020996968	56075605906890	256075605906890	Σ_{14}
Σ_{16}	2	131074	86224516	8676159108	313851940358	5956071755270	72421932894472	635371886315784	4341412264019466	24341412264019466	Σ_{16}

146. **Table of these Formulae.**—By substituting particular numbers for k and n and evaluating the determinants, we obtain the following table of the graduation formulae which are obtained by fitting a polynomial to the data.

Case I. $k=0$, i.e. fitting a straight line to the data.

$$u_0' = \frac{1}{2n+1}\{u_0 + (u_1 + u_{-1}) + (u_2 + u_{-2}) + \ldots + (u_n + u_{-n})\}.$$

Case II. $k=1$, i.e. fitting a parabola of degree 2 or 3. The general formula in this case is

$$u_0' = p_n u_n + p_{n-1} u_{n-1} + \ldots + p_{-n} u_{-n},$$

where
$$p_s = 3 \cdot \frac{3n^2 + 3n - 1 - 5s^2}{(2n-1)(2n+1)(2n+3)}.$$

This gives

$n=1$ $\quad u_0' = u_0.$

$n=2$ $\quad u_0' = \frac{1}{35}\{17u_0 + 12(u_1 + u_{-1}) - 3(u_2 + u_{-2})\} = u_0 - \frac{3}{35}\Delta^4 u_{-2}.$

$n=3$ $\quad u_0' = \frac{1}{21}\{7u_0 + 6(u_1 + u_{-1}) + 3(u_2 + u_{-2}) - 2(u_3 + u_{-3})\}.$

$n=4$ $\quad u_0' = \frac{1}{231}\{59u_0 + 54(u_1 + u_{-1}) + 39(u_2 + u_{-2}) + 14(u_3 + u_{-3})$
$\qquad - 21(u_4 + u_{-4})\}.$

$n=5$ $\quad u_0' = \frac{1}{429}\{89u_0 + 84(u_1 + u_{-1}) + 69(u_2 + u_{-2}) + 44(u_3 + u_{-3})$
$\qquad + 9(u_4 + u_{-4}) - 36(u_5 + u_{-5})\}.$

$n=6$ $\quad u_0' = \frac{1}{143}\{25u_0 + 24(u_1 + u_{-1}) + 21(u_2 + u_{-2}) + 16(u_3 + u_{-3})$
$\qquad + 9(u_4 + u_{-4}) \qquad - 11(u_6 + u_{-6})\}.$

$n=7$ $\quad u_0' = \frac{1}{1105}\{167u_0 + 162(u_1 + u_{-1}) + 147(u_2 + u_{-2})$
$\qquad + 122(u_3 + u_{-3}) + 87(u_4 + u_{-4}) + 42(u_5 + u_{-5})$
$\qquad - 13(u_6 + u_{-6}) - 78(u_7 + u_{-7})\}.$

$n=8$ $\quad u_0' = \frac{1}{323}\{43u_0 + 42(u_1 + u_{-1}) + 39(u_2 + u_{-2}) + 34(u_3 + u_{-3})$
$\qquad + 27(u_4 + u_{-4}) + 18(u_5 + u_{-5}) + 7(u_6 + u_{-6})$
$\qquad - 6(u_7 + u_{-7}) - 21(u_8 + u_{-8})\}.$

$n=9$ $\quad u_0' = \frac{1}{2261}\{269u_0 + 264(u_1 + u_{-1}) + 249(u_2 + u_{-2})$
$\qquad + 224(u_3 + u_{-3}) + 189(u_4 + u_{-4}) + 144(u_5 + u_{-5})$
$\qquad + 89(u_6 + u_{-6}) + 24(u_7 + u_{-7}) - 51(u_8 + u_{-8})$
$\qquad - 136(u_9 + u_{-9})\}.$

$n=10$ $\quad u_0' = \frac{1}{3059}\{329u_0 + 324(u_1 + u_{-1}) + 309(u_2 + u_{-2})$
$\qquad + 284(u_3 + u_{-3}) + 249(u_4 + u_{-4}) + 204(u_5 + u_{-5})$
$\qquad + 149(u_6 + u_{-6}) + 84(u_7 + u_{-7}) + 9(u_8 + u_{-8})$
$\qquad - 76(u_9 + u_{-9}) - 171(u_{10} + u_{-10})\}.$

Case III. $k=2$, *i.e. fitting a parabola of degree* 4 *or* 5.

$n=2 \quad u_0' = u_0.$

$$n=3 \quad u_0' = \tfrac{1}{231}\{131u_0 + 75(u_1+u_{-1}) - 30(u_2+u_{-2}) + 5(u_3+u_{-3})\}.$$

$$n=4 \quad u_0' = \tfrac{1}{429}\{179u_0 + 135(u_1+u_{-1}) + 30(u_2+u_{-2}) - 55(u_3+u_{-3}) + 15(u_4+u_{-4})\}.$$

$$n=5 \quad u_0' = \tfrac{1}{429}\{143u_0 + 120(u_1+u_{-1}) + 60(u_2+u_{-2}) - 10(u_3+u_{-3}) - 45(u_4+u_{-4}) + 18(u_5+u_{-5})\}.$$

$$n=6 \quad u_0' = \tfrac{1}{2431}\{677u_0 + 600(u_1+u_{-1}) + 390(u_2+u_{-2}) + 110(u_3+u_{-3}) - 135(u_4+u_{-4}) - 198(u_5+u_{-5}) + 110(u_6+u_{-6})\}.$$

$$n=7 \quad u_0' = \tfrac{1}{46189}\{11063u_0 + 10125(u_1+u_{-1}) + 7500(u_2+u_{-2}) + 3755(u_3+u_{-3}) - 165(u_4+u_{-4}) - 2937(u_5+u_{-5}) - 2860(u_6+u_{-6}) + 2145(u_7+u_{-7})\}.$$

$$n=8 \quad u_0' = \tfrac{1}{4199}\{883u_0 + 825(u_1+u_{-1}) + 660(u_2+u_{-2}) + 415(u_3+u_{-3}) + 135(u_4+u_{-4}) - 117(u_5+u_{-5}) - 260(u_6+u_{-6}) - 195(u_7+u_{-7}) + 195(u_8+u_{-8})\}.$$

$$n=9 \quad u_0' = \tfrac{1}{7429}\{1393u_0 + 1320(u_1+u_{-1}) + 1110(u_2+u_{-2}) + 790(u_3+u_{-3}) + 405(u_4+u_{-4}) + 18(u_5+u_{-5}) - 290(u_6+u_{-6}) - 420(u_7+u_{-7}) - 255(u_8+u_{-8}) + 340(u_9+u_{-9})\}.$$

$$n=10 \quad u_0' = \tfrac{1}{260015}\{44003u_0 + 42120(u_1+u_{-1}) + 36660(u_2+u_{-2}) + 28190(u_3+u_{-3}) + 17655(u_4+u_{-4}) + 6378(u_5+u_{-5}) - 3940(u_6+u_{-6}) - 11220(u_7+u_{-7}) - 13005(u_8+u_{-8}) - 6460(u_9+u_{-9}) + 11628(u_{10}+u_{-10})\}.$$

147. Selection of the Appropriate Formula.—Among the many formulae of the last section, we have to determine the one which is most appropriate to the particular material that is to be graduated; this may be done in the following way:

From the formulae of § 145 we see that if we tried to fit an ordinary parabola $y=c_0+c_1x+c_2x^2$ to data $u_{-n}, u_{-n+1}, \ldots, u_n$, then we should have

$$c_0\Sigma_0 + c_2\Sigma_2 = M_0,$$
$$c_0\Sigma_2 + c_2\Sigma_4 = M_2,$$

and therefore
$$c_2 = \frac{\Sigma_0 M_2 - \Sigma_2 M_0}{\Sigma_0\Sigma_4 - \Sigma_2^2}.$$

Now c_2 is $\frac{1}{2}\Delta^2 y$, where $\Delta^2 y$ denotes the second difference of y, which is of course constant since y is of the second degree in x. So if we try to represent a certain stretch of the data, say from u_{-n} to u_n inclusive, by an ordinary parabola, then the most probable value of $\Delta^2 y$ for this parabola is

$$\frac{2(\Sigma_0 M_2 - \Sigma_2 M_0)}{\Sigma_0 \Sigma_4 - \Sigma_2^2}.$$

This is readily calculated (the Σ's being given at once by the table of § 145), and thus we can make a preliminary test of the value of $\Delta^2 y$ for curves roughly fitting the data in different "stretches." This enables us to judge what is the lowest order of parabola which will give a satisfactory fit when we use some definite number $(2n+1)$ of data in the graduation formula.

Ex. 1.—*If a set of observations* $y_1, y_2, \ldots, y_n$, *corresponding to equidistant values of the argument* x, *is given, and if we represent these as well as possible by a formula of the type* $y = Ax + B$, *so that* Δy *is constant for the graduated values, then the most probable value of this constant* Δy *is*

$$\frac{6}{n(n^2-1^2)}\{1.(n-1)\Delta y_1 + 2.(n-2)\Delta y_2 + 3.(n-3)\Delta y_3 + \ldots + (n-1).1.\Delta y_{n-1}\}.$$

Ex. 2.—*If a set of observations* $y_1, y_2, \ldots, y_n$, *corresponding to equidistant values of the argument* x, *is given, and if we represent these as well as possible by a formula of the type* $y = Ax^2 + Bx + C$, *so that* $\Delta^2 y$ *is constant for the graduated values, then the most probable value of this constant* $\Delta^2 y$ *is*

$$\frac{30}{n(n^2-1^2)(n^2-2^2)}\sum_p \{p(p+1)(n-p)(n-p-1)\Delta^2 y_p\}.$$

Ex. 3.—*If a set of observations* $y_1, y_2, \ldots, y_n$, *corresponding to equidistant values of the argument* x, *is given, and if we represent these as well as possible by a formula of the type* $y = Ax^3 + Bx^2 + Cx + D$, *so that* $\Delta^3 y$ *is constant for the graduated values, then the most probable value of this constant is*

$$\frac{140}{n(n^2-1^2)(n^2-2^2)(n^2-3^2)} \sum_p \{p(p+1)(p+2)(n-p)(n-p-1)(n-p-2)\Delta^3 y_p\}.$$

148. **Tests performed on Actual Data.***—We shall now consider the relative merits of Summation formulae and Least-square formulae as tested by their performance when applied to definite numerical data.

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* Sherriff, *loc. cit.*

We have first to decide what is to be accepted as the measure of good performance in a graduation. The test we shall use may be described thus:

Consider some known analytic function of x, such as $\log x$, of which tables accurate to, say, 6 places are available. A 4-place table of this function may be prepared by omitting the last two digits (which will be called the *tail*) and "*forcing*," *i.e.* increasing the last retained digit by unity when the omitted tail begins with one of the digits 5, 6, 7, 8, or 9. We can regard the values of $\log x$ given by the 4-place table as affected with "errors," namely, the errors which have been produced by omitting the tails. Let us now take a sequence of these 4-place values, and graduate them by the graduation formula which is to be tested; the effect of the graduation should be to smooth out the "errors" and restore, to some extent at least, the more accurate values of the 6-place table. The success with which this is performed may be taken as a measure of the merit of the graduation formula; for it must be remembered that the purpose of a graduation formula is precisely to reduce the magnitude of accidental errors. The advantage of using a known function, such as $\log x$, for the test is that we can be certain that the errors (viz. the tails) are accidental, *i.e.* non-systematic. There is, however, the disadvantage that the errors do not obey the normal law of frequency, since within the limits $\pm 0{\cdot}5$ of the last place, the probability of an error ϵ does not vary with ϵ.

In the following table this method of testing is applied to the function $\frac{10^7}{x} - 39{,}999{\cdot}95$. Spencer's formula and the Least-square formula $k = 1$, $m = 10$ are used. The merits of the graduated values are obtained by comparing columns 9, 10, and 11: the result is that the sum of the squares of the residual errors is 873 when the Least-square formula is used, and 1327 when Spencer's formula is used.

[TABLE

GRADUATION OF THE RECIPROCALS OF NUMBERS BY SPENCER'S FORMULA AND THE LEAST-SQUARE FORMULA $k=1$, $n=10$

No. x.	$10^7/x-39{,}999{\cdot}95=\mathrm{U}$.	Forced Value of $\mathrm{U}=u$.	Graduated Value of $u=u'$.		Difference between Ungraduated and Graduated and True Values × 10^2.			$(\mathrm{U}-u)^2$. (Col. 6)^2.	$(\mathrm{U}-u')^2$.	
			By Least-square Formula $k=1$, $n=10$.	By Spencer's Formula.	Col. 2-Col. 3.	Col. 2-Col. 4.	Col. 2-Col. 5.		(Col. 7)^2.	(Col. 8)^2.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
201	9751·29	9751	...	...	...	...	...	...	...	...
202	9505·00	9505	...	...	...	...	...	...	...	...
203	9261·13	9261	...	...	...	...	...	...	...	...
204	9019·66	9020	...	...	...	...	...	...	...	...
205	8780·54	8781	...	...	...	...	...	...	...	...
206	8543·74	8544	...	...	...	.	...	..	...	...
207	8309·23	8309	...	...	...	...	...	...	...	...
208	8076·97	8077	...	...	...	...	...	...	...	...
209	7846·94	7847	...	...	...	...	...	...	...	...
210	7619·10	7619	...	...	...	...	...	...	...	...
211	7393·41	7393	7393·36	7393·29	41	5	12	1681	25	144
212	7169·86	7170	7169·80	7169·73	−14	6	13	196	36	169
213	6948·41	6948	6948·34	6948·26	41	7	15	1681	49	225
214	6729·02	6729	6728·93	6728·94	2	9	8	4	81	64
215	6511·68	6512	6511·63	6511·64	−32	5	4	1024	25	16
216	6296·35	6296	6296·31	6296·36	35	4	−1	1225	16	1
217	6083·00	6083	6082·99	6083·05	0	1	−5	0	1	25
218	5871·61	5872	5871·58	5871·69	−39	3	−8	1521	9	64
219	5662·15	5662	5662·15	5662·23	15	0	−8	225	0	64
220	5454·60	5455	5454·64	5454·67	−40	−4	−7	1600	16	49
221	5248·92	5249	5248·97	5248·97	−8	−5	−5	64	25	25
222	5045·10	5045	5045·14	5045·11	10	−4	−1	100	16	1
223	4843·10	4843	4843·10	4843·09	10	0	1	100	0	1
224	4642·91	4643	4642·88	4642·88	−9	3	3	81	9	9
225	4444·49	4444	4444·46	4444·46	49	3	3	2401	9	9
226	4247·84	4248	4247·80	4247·79	−16	4	5	256	16	25
227	4052·91	4053	4052·84	4052·86	−9	7	5	81	49	25
228	3859·70	3860	3859·59	3859·64	−30	11	6	900	121	36
229	3668·17	3668	3668·08	3668·10	17	9	7	289	81	49
230	3478·31	3478	3478·19	3478·23	31	12	8	961	144	64
231	3290·09	3290	3289·99	3290·00	9	10	9	81	100	81
232	3103·50	3103	3103·44	3103·40	50	6	10	2500	36	100
233	2918·50	2919	2918·47	2918·41	−50	3	9	2500	9	81
234	2735·09	2735	...	...	...	...	...	...	...	...
235	2553·24	2553	...	...	...	...	...	..	..	...
236	2372·93	2373	...	...	...	...	...	...	...	...
237	2194·14	2194	...	...	...	...	...	...	...	...
238	2016·86	2017	...	...	...	...	...	...	...	...
239	1841·05	1841	...	...	...	...	...	...	...	...
240	1666·72	1667	...	...	...	...	...	...	...	...
241	1493·83	1494	...	...	...	...	..	...	...	...
242	1322·36	1322	...	...	...	...	...	...	...	...
243	1152·31	1152	...	...	...	...	...	...	...	...
Total for columns 9, 10, and 11 . .								19471	873	1327

149. Graduation by Reduction of Probable Error.—The graduation formulae which we have obtained by fitting polynomials have been derived otherwise by W. F. Sheppard,[*] who approaches the problem in the following way:

As before, let $\ldots u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$ be the ungraduated values of u. Suppose that the determination of each of these is subject to the probable error ϵ. If the graduated value of u_0 is

$$u_0' = p_0u_0 + p_1u_1 + p_2u_2 + \ldots + p_nu_n \\ + p_{-1}u_{-1} + p_{-2}u_{-2} + \ldots + p_{-n}u_{-n},$$

then the probable error of u_0' is

$$(p_{-n}^2 + p_{-n+1}^2 + \ldots + p_{n-1}^2 + p_n^2)^{\frac{1}{2}}\epsilon.$$

Sheppard lays down the condition that *this quantity is to be a minimum,* subject to a further condition which secures that the graduated values shall not differ systematically from the ungraduated. This latter condition he takes in the form that *u_0' is to differ from u_0 only by differences of u_0 of order $(j+1)$ and upwards*; this amounts to supposing that the $(j+1)$th differences of the u's are negligible.

Now, by a discussion resembling Laplace's and Gauss's *Theoria Combinationis* proof of the Method of Least Squares (§ 115), we see that Sheppard's conditions are really equivalent to the two conditions which were laid down in § 145; and hence *the graduation formulae obtained by this method are identical with those which have been obtained* (§§ 145-147) *by fitting polynomials to the data by Least Squares.*

150. The Method of Interlaced Parabolas.—A method of graduation proposed in 1922,[†] which yields satisfactory results when applied to actuarial data, may be explained as follows.

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[*] *Proc. of the Fifth Int. Congress of Mathematicians* (Cambridge, 1912), ii. p. 348, and other papers quoted in the footnote, p. 291. A. C. Aitken, *Proc. Roy. Soc. Edin.* **53** (1932), p. 54, has solved the problem of polynomial graduation by use of orthogonal polynomials, obtaining a process which is decidedly preferable to those of Sheppard and Miss Sherriff.

[†] *J.I.A.* *53* (1922), p. 92.

Let a polynomial of the third degree

$$u = a + bx + cx^2 + dx^3 \tag{1}$$

be determined by the conditions that (1) it is to take as nearly as possible the values $u_{-m}, u_{-m+1}, \ldots, u_{-2}, u_0, u_2, \ldots, u_m$, when x has the values $-m, -m+1, \ldots, -2, 0, 2, \ldots, m$ respectively, and (2) it is to take precisely the values u'_{-1} and u_1' when x has the values -1 and 1 respectively. Here as usual $u_0, u_1, \ldots$ denote ungraduated values and $u_0', u_1', \ldots$ denote graduated values, so that we are really finding a parabola of the third degree which will fit as well as possible the ungraduated data $u_{-m}, u_{-m+1}, \ldots, u_m$, and will also fit rigorously the two graduated values u'_{-1} and u_1', which for the moment are supposed already known. Then *the ordinate of this parabola at* $x = 0$ *will be taken to be the graduated value* u_0'. The graduated values will therefore be given by a series of interlaced parabolas, each of which passes through three consecutive graduated values, so that each successive pair of these parabolas has two points on the graduated curve in common. The equations of condition are evidently

$$u_r = a + br + cr^2 + dr^3, \tag{2}$$

for $r = -m, -m+1, \ldots, -2, 0, 2, \ldots, m$; and we may without error include $r = -1$ and $r = 1$ in this sequence, since the effect of the two equations of condition thus introduced will be nullified by the two equations which have to be satisfied rigorously, namely,

$$\left.\begin{aligned} u'_{-1} &= a - b + c - d \\ u_1' &= a + b + c + d \end{aligned}\right\} \tag{3}$$

Denoting as usual $\sum\limits_{r=-m}^{r=m} r^p$ by Σ_p, we have therefore to choose a, b, c, d so as to make

$$\sum_{r=-m}^{m} (a + br + cr^2 + dr^3 - u_r)^2$$

or

$$a^2\Sigma_0 + b^2\Sigma_2 + c^2\Sigma_4 + d^2\Sigma_6 + 2ac\Sigma_2 + 2bd\Sigma_4 - 2a\Sigma u_r - 2b\Sigma ru_r$$
$$- 2c\Sigma r^2u_r - 2d\Sigma r^3u_r$$

a minimum subject to the rigorous equations (3). The normal equations are therefore (§ 108)

$$\left.\begin{aligned} a\Sigma_0 + c\Sigma_2 - \Sigma u_r + \lambda + \mu &= 0 \\ b\Sigma_2 + d\Sigma_4 - \Sigma r u_r - \lambda + \mu &= 0 \\ c\Sigma_4 + a\Sigma_2 - \Sigma r^2 u_r + \lambda + \mu &= 0 \\ d\Sigma_6 + b\Sigma_4 - \Sigma r^3 u_r - \lambda + \mu &= 0 \end{aligned}\right\} . \qquad (4)$$

The unknowns a, b, c, d, λ, μ are to be determined from equations (3) and (4). Evidently we need only consider the equations which involve a, c, and $(\lambda+\mu)$, which are

$$u'_{-1} + u_1' = 2a + 2c,$$

$$a\Sigma_0 + c\Sigma_2 - \Sigma u_r + (\lambda+\mu) = 0,$$

$$c\Sigma_4 + a\Sigma_2 - \Sigma r^2 u_r + (\lambda+\mu) = 0,$$

and we have also $\quad u_0' = a.$

Eliminating a, c, and $(\lambda+\mu)$ from these four equations, we have

$$u_1' + u'_{-1} = \frac{2(\Sigma_4 - 2\Sigma_2 + \Sigma_0)}{\Sigma_4 - \Sigma_2} u_0' + \frac{2\Sigma(r^2-1)u_r}{\Sigma_4 - \Sigma_2}.$$

Thus if we write $\cos\theta$ for $\dfrac{\Sigma_4 - 2\Sigma_2 + \Sigma_0}{\Sigma_4 - \Sigma_2}$, and K_x for $\dfrac{2\Sigma(r^2-1)u_{x+r}}{\Sigma_4 - \Sigma_2}$, the graduated values of u_x' satisfy the linear difference equation

$$u'_{x+1} - 2\cos\theta\, u_x' + u'_{x-1} = K_x,$$

the solution of which is

$$u_x' = A\cos x\theta + B\sin x\theta + \frac{K_0 \sin x\theta + K_1 \sin(x-1)\theta + \ldots + K_{x-1}\sin\theta}{\sin\theta},$$

where A and B are the constants of integration; they may be used, in the case of mortality data, to make the deviations of the actual from the expected deaths, and the accumulated deviations, zero; or in all cases they may be used to reproduce the moments of order zero and one, of the ungraduated u's.

151. A Method of Graduation based on Probability.[*]—The methods described above fulfil their purpose of smoothing out irregularities from observational data in a way which is on the whole efficient. From the purely theoretical standpoint they are not altogether satisfactory, since each of them contains arbitrary or empirical elements whose introduction does not appear to be logically necessary; *e.g.* in the methods of §§ 142, 150 it is not obvious (apart from mere convenience) why we should fit parabolic curves to the observations rather than fit curves such as (say)

$$y = ae^{-\gamma x}\left(1+\frac{x}{c}\right)^{\lambda}.$$

In order to find a sounder basis for the theory, we must remember that the problem of graduation belongs essentially to the mathematical theory of probability;[†] we have the given observations, and they would constitute the "most probable" values of u for the corresponding values of the argument, were it not that we have *a priori* grounds for believing that the true values of u form a smooth sequence, the irregularities being due to accidental causes which it is desirable to eliminate. The problem is to combine all the materials of judgment—the observed values and the *a priori* considerations—in order to obtain the "most probable" values of u.

Let us then suppose that we are concerned with a number u_x which depends on an argument x, and suppose that we have n data which are affected with uncertainties or irregularities due, *e.g.*, to accidental errors of observation; so that when u_x is plotted as a function of x, the n points so obtained do not lie on a smooth curve, although there is a strong antecedent probability that if the observations had been more accurate the curve would have been smooth. We may make the somewhat vague word "smooth" more precise by interpreting it to mean, *e.g.*, that the third differences $\Delta^3 u_x$ are to be very small.

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[*] Whittaker, *Proc. Edin. Math. Soc.* **41**, p. 63 (read Nov. 14, 1919: printed, with additions, in the volume for 1922–23). The method has been further improved by Whittaker, *Proc. R.S. Edin.* (1924).

[†] The first recognition of this fundamental principle seems to have been made by Mr. G. King in the course of the discussion on Dr. T. B. Sprague's paper of 1886, *J.I.A.* **26**, p. 77: "What is the real object of graduation? Many would reply, to get a smooth curve; but that is not quite correct. The reply should be, to get the most probable deaths."

Now consider the following hypothesis: that the true value, which should have been obtained by the observation u_1, lies between u_1' and $u_1' + \sigma$, where σ is a small constant number (*e.g.* one unit in the last decimal place used in the measures); that the true value which should have been obtained by the observation for u_2 lies between u_2' and $u_2' + \sigma$, and so on; and finally the true value which should have been obtained by the observation for u_n lies between u_n' and $u_n' + \sigma$. This hypothesis we shall call "*hypothesis H.*" Before the observations have been made we have nothing to guide us as to the probability of this hypothesis H except the degree of smoothness of the sequence $u_1', \ldots u_n'$, which may be measured by the smallness of the sum of the squares of the third differences.

$$S = (u_4' - 3u_3' + 3u_2' - u_1')^2 + (u_5' - 3u_4' + 3u_3' - u_2')^2 + \ldots + (u_n' - 3u_{n-1}' + 3u_{n-2}' - u_{n-3}')^2.$$

S may be called the *measure of roughness* of the sequence.

The theory may be extended to the case when the observations are not taken at equidistant values of the argument, by taking instead of S the sum of the squares of the third *divided* differences of the graduated values.

We may therefore, by analogy with the normal law of frequency, suppose that the *a priori* probability of hypothesis H is

$$ce^{-\lambda^2 S}\sigma^n, \tag{A}$$

where c and λ denote constants.

Next let us consider the *a priori* probability that the measures obtained by the observations will be $u_1, u_2, \ldots, u_n$, on the assumption that hypothesis H is true.

Since the true value of the first observed quantity is, on this hypothesis, u_1', the probability that a value between u_1 and $u_1 + \sigma$ will actually be observed is (postulating the normal law of error)

$$\frac{h_1}{\sqrt{\pi}} e^{-h_1^2(u_1 - u_1')^2}\sigma,$$

where h_1 is a constant which measures the precision with which this observation can be made. Similarly the probability that a value between u_2 and $u_2+\sigma$ will actually be obtained for the second observed measure is

$$\frac{h_2}{\sqrt{\pi}}e^{-h_2^2(u_2-u_2')^2}\sigma,$$

where h_2 is the measure of precision of this observation. Thus on the assumption that hypothesis H is true, the *a priori* probability that the observed measure of the first observed quantity will lie between u_1 and $u_1+\sigma$, the observed measure of the second observed quantity between u_2 and $u_2+\sigma$, and so on, is

$$\frac{h_1 h_2 \ldots h_n}{(\sqrt{\pi})^n}e^{-\mathrm{F}}\sigma^n, \tag{B}$$

where F denotes the sum

$$\mathrm{F} \equiv h_1^2(u_1-u_1')^2+h_2^2(u_2-u_2')^2+\ldots+h_n^2(u_n-u_n')^2.$$

The sums S and F enable us to express numerically the *smoothness* of the graduated values, and the *fidelity* of the graduated to the ungraduated values respectively.

We must now make use of the fundamental theorem in the theory of Inductive Probability, which is as follows. Suppose that a certain observed phenomenon may be accounted for by any one of a certain number of hypotheses, of which one, and not more than one, must be true: suppose, moreover, that the probability of the sth hypothesis, as based on information in our possession before the phenomenon is observed, is p_s, while the probability of the observed phenomenon, on the assumption of the truth of the sth hypothesis, is P_s. Then when the observation of the phenomenon is taken into consideration, the probability of the sth hypothesis is

$$\frac{p_s\mathrm{P}_s}{\Sigma p_s\mathrm{P}_s},$$

where the symbol Σ denotes the summation over all the hypotheses. It follows from this that whereas *before* the phenomenon was observed the most probable hypothesis was that for which p_s was greatest, the most probable hypothesis *after* the phenomenon has been observed is that for which the product $\mathrm{P}_s p_s$ is greatest. Applying this theorem to the case under consideration and combining expressions (A) and (B), we see that the most probable hypothesis is that for which

$$\frac{ch_1 h_2 \ . \ . \ . \ h_n}{(\sqrt{\pi})^n} e^{-\lambda^2 S - F_{\sigma}{}^{2n}}$$

is a maximum; that is to say, *the most probable set* $u_1' \ . \ . \ . \ u_n'$ *of values of the quantities is that which makes*

$$\lambda^2 S + F$$

a minimum.

152. **The Analytical Formulation.**—Writing down the ordinary conditions for a minimum, we obtain the equations

$$\begin{aligned}
h_1^2 u_1 &= h_1^2 u_1' - \lambda^2 \Delta^3 u_1', \\
h_2^2 u_2 &= h_2^2 u_2' + 3\lambda^2 \Delta^3 u_1' - \lambda^2 \Delta^3 u_2', \\
h_3^2 u_3 &= h_3^2 u_3' - 3\lambda^2 \Delta^3 u_1' + 3\lambda^2 \Delta^3 u_2' - \lambda^2 \Delta^3 u_3', \\
h_4^2 u_4 &= h_4^2 u_4' + \lambda^2 \Delta^3 u_1' - 3\lambda^2 \Delta^3 u_2' + 3\lambda^2 \Delta^3 u_3' - \lambda^2 \Delta^3 u_4', \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
h_n^2 u_n &= h_n^2 u_n' + \lambda^2 \Delta^3 u'_{n-3}.
\end{aligned}$$

We shall now make the simplifying assumption that the measure of precision is the same for all the data, so $h_1 = h_2 = \ . \ . \ . = h_n$.

If this is not the case, we graduate some function of u, such as $\log u$, instead of u, choosing this function so that its measure of precision has nearly the same value for all values of the argument.

If we write $h_1^2 = \epsilon\lambda^2$, the equations may now be written

$$\left.\begin{aligned}
\epsilon u_1 &= \epsilon u_1' - \Delta^3 u_1' \\
\epsilon u_2 &= \epsilon u_2' + 3\Delta^3 u_1' - \Delta^3 u_2' \\
\epsilon u_3 &= \epsilon u_3' - 3\Delta^3 u_1' + 3\Delta^3 u_2' - \Delta^3 u'_3 \\
\epsilon u_4 &= \epsilon u_4' + \Delta^3 u_1' - 3\Delta^3 u_2' + 3\Delta^3 u_3' - \Delta^3 u_4' \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\epsilon u_n &= \epsilon u_n' + \Delta^3 u'_{n-3}
\end{aligned}\right\} . \qquad (1)$$

Now all these equations, except the three first and the three last, are of the form

$$\epsilon u_x = \epsilon u_x' - \Delta^6 u'_{x-3}.$$

Moreover, if we introduce a quantity u_0' such that $\Delta^3 u_0' = 0$, the third equation becomes

$$\epsilon u_3 = \epsilon u_3' - \Delta^6 u_0',$$

which is of the same form; and similarly the first two and last three equations can be brought to the same form by introducing new quantities u'_{-1}, u'_{-2}, u'_{n+1}, u'_{n+2}, u'_{n+3}, such that

$$\Delta^3 u'_{-1}=0,\ \Delta^3 u'_{-2}=0,\ \Delta^3 u'_{n-2}=0,\ \Delta^3 u'_{n-1}=0,\ \Delta^3 u_n'=0.$$

Thus *the graduated values* u_x' *satisfy the linear difference equation*

$$\epsilon u_x' - \Delta^6 u'_{x-3} = \epsilon u_x, \tag{2}$$

being in fact the particular solution of this equation which satisfies the six terminal conditions

$$\Delta^3 u_0'=0,\ \Delta^3 u'_{-1}=0,\ \Delta^3 u'_{-2}=0,\ \Delta^3 u'_{n-2}=0,\ \Delta^3 u'_{n-1}=0,\ \Delta^3 u_n'=0, \tag{3}$$

whence we have at once

$$\Delta^4 u'_{-2}=0,\ \Delta^4 u'_{-1}=0,\ \Delta^5 u'_{-2}=0,\ \Delta^4 u'_{n-2}=0,\ \Delta^4 u'_{n-1}=0,\ \Delta^5 u'_{n-2}=0. \tag{4}$$

153. The Theorems of Conservation.—From (2) we have by summation

$$\begin{aligned}\epsilon(u_1'+u_2'+\ldots+u_n')-\epsilon(u_1+u_2+\ldots+u_n)&=\Delta^6 u'_{-2}+\Delta^6 u'_{-1}+\ldots+\Delta^6 u'_{n-3}\\&=\Delta^5 u'_{n-2}-\Delta^5 u'_{-2}\\&=0,\text{ by (4).}\end{aligned}$$

Therefore

$$u_1'+u_2'+\ldots+u_n'=u_1+u_2+\ldots+u_n. \tag{5}$$

Moreover, by (2),

$$\begin{aligned}\epsilon(u_1'+2u_2'+\ldots+nu_n')-\epsilon(u_1+2u_2+\ldots+nu_n)&=\Delta^6 u'_{-2}+2\Delta^6 u'_{-1}+\ldots+n\Delta^6 u'_{n-3}\\&=n\Delta^5 u'_{n-2}-\Delta^4 u'_{n-2}+\Delta^4 u'_{-2}\\&=0,\text{ by (4).}\end{aligned}$$

Therefore

$$u_1'+2u_2'+\ldots+nu_n'=u_1+2u_2+\ldots+nu_n. \tag{6}$$

Next, by (2),

$$\begin{aligned}&\epsilon(u_1'+2^2u_2'+\ldots+n^2u_n')-\epsilon(u_1+2^2u_2+\ldots+n^2u_n)\\&\quad=\Delta^6 u'_{-2}+2^2\Delta^6 u'_{-1}+\ldots+n^2\Delta^6 u'_{n-3}\\&\quad=n^2\Delta^5 u'_{n-2}-(2n-1)\Delta^4 u'_{n-2}+2\Delta^3 u'_{n-2}-\Delta^3 u'_{-2}-\Delta^3 u'_{-1}\\&\quad=0,\text{ by (3) and (4).}\end{aligned}$$

Therefore

$$u_1'+2^2u_2'+\ldots+n^2u_n'=u_1+2^2u_2+\ldots+n^2u_n. \tag{7}$$

Equations (5), (6), (7) show that *the moments of orders* 0, 1, 2 *are the same for the graduated data as for the original data.* This may be called the *Theorem of the Conservation of Moments.* We may express it by saying that *the graph which represents the ungraduated data and the graph which represents the graduated data have the same area, the same* x *co-ordinate of the centre of gravity, and the same moment of inertia about any line parallel to the axis of* u.

154. **The Solution of the Difference Equation.***—We have now to solve the central difference equation of graduation,

$$\epsilon u'_x - \Delta^6 u'_{x-3} = \epsilon u_x,$$

subject to the six terminal conditions imposed in § 152. Let us assume for the present that ϵ is given. Then we may obtain a general solution by means of symbolic operators as follows.

Since $\Delta = \mathrm{E} - 1$, the equation, in terms of E, becomes

$$\left[\frac{(\mathrm{E}-1)^6 - \epsilon \mathrm{E}^3}{\mathrm{E}^3}\right] u'_x = -\epsilon u_x,$$

so that

$$u'_x = -\epsilon \left[\frac{\mathrm{E}^3}{(\mathrm{E}-1)^6 - \epsilon \mathrm{E}^3}\right] u_x. \tag{8}$$

Considerations of symmetry lead us to expect that each u' will be given as a linear function of u's, with coefficients symmetrically placed about that of the central term. This suggests that we expand the operator on the right of equation (8) in powers of both E and E^{-1}, in fact as a *Laurent series.*† Such an expansion is readily seen to be possible, for the equation $(z-1)^6 - \epsilon z^3 = 0$, being a reciprocal equation, has its six roots reciprocal in pairs, so that three are less in modulus than unity, and the remaining three greater; hence $[(z-1)^6 - \epsilon z^3]^{-1}$ may be expanded as a Laurent series, convergent for $z = 1$, and the coefficients, with which we are chiefly concerned, may be found by the usual theory. That these coefficients will be symmetrically disposed follows from the fact that the operator is left unaltered by the substitution of E^{-1} for E.

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* The method of solution described in §§ 154–155 is due to Dr. A. C. Aitken and was first given in a thesis for the doctorate of the University of Edinburgh, submitted in May, 1925.

† Cf. Whittaker and Watson, *Modern Analysis*, § 5.6.

Appeal to the theory of Functions of a Complex Variable may be avoided, if preferred, by having recourse to the theory of Partial Fractions. If we resolve the operator in question into six partial operators by this theory, then, for the reasons already stated, three of the partial operators may be expanded in powers of E, three in powers of E^{-1}, whence by addition the complete expansion is obtained.

Either of these ways leads to the same graduating formula, namely,

$$u'_x = k_0 u_x + k_1(u_{x+1} + u_{x-1}) + k_2(u_{x+2} + u_{x-2}) + \ldots, \tag{9}$$

where $k_n = -\epsilon\Sigma\dfrac{a^{n+2}}{(a-\beta)(a-\gamma)(a-a^{-1})(a-\beta^{-1})(a-\gamma^{-1})}$, (10)

in which a, β, γ are the three roots less in modulus than unity* of the equation $(z-1)^6 - \epsilon z^3 = 0$, and Σ denotes interchange of a, β, γ followed by summation. One of these roots is evidently real; hence we may write them as r_1, $r_2e^{i\theta}$, $r_2e^{-i\theta}$, when (10) becomes

$$k_n = \frac{\epsilon r_1^2}{\left(r_1^2 - 2r_1r_2\cos\theta + r_2^2\right)\left(r_1^2 - 2\frac{r_1}{r_2}\cos\theta + \frac{1}{r_2^2}\right)}\left\{A + \frac{B}{C}\right\}, \tag{11}$$

where $A = \dfrac{r_1^{n+1}}{1-r_1^2}$,

$$B = r_2^{n+1}\left[r_2^2\sin(n-2)\theta - r_2\left(r_1 + \frac{1}{r_1}\right)\sin(n-1)\theta + \frac{1}{r_2}\left(r_1 + \frac{1}{r_1}\right)\sin(n+1)\theta - \frac{1}{r_2^2}\sin(n+2)\theta\right],$$

$$C = (1 - r_2^2)\sin\theta\left(r_2^2 - 2\cos 2\theta + \frac{1}{r_2^2}\right).$$

By means of (11) the coefficients k_n have been calculated for a number of representative values of ϵ, 0·01, 0·02, 0·05, 0·1, 0·25, and 1. A table of these to four decimal places is given on p. 314.

A defect inherent in most methods of graduation based on linear compounding of u's on either side of a particular u is that *it is impossible to graduate the whole of a given set of*

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[*] It is assumed throughout that ϵ is positive, not zero.

data; e.g. Spencer's formula is inapplicable when we are within ten places of either terminal. (See the example of § 144, p. 291.) This defect will also seriously impair the present method, unless by some means, without departing from the conditions of the problem, we can attach to each end of a given set the additional, or, as we may call them, *auxiliary* data, $u_{n+1}, u_{n+2}, \ldots$, and $u_0, u_{-1}, \ldots$, the existence of which is implied in formula (9). (The number of data to be attached in practice will depend on the degree of accuracy required and the rapidity with which the successive graduating coefficients diminish.) As in § 152 we introduced six new quantities $u'_{n+1}, u'_{n+2}, u'_{n+3}, u'_0, u'_{-1}, u'_{-2}$, so let us now introduce a further indefinite number $u'_{n+4}, u'_{n+5}, \ldots$, and $u'_{-3}, u'_{-4}, \ldots$, under the same condition, namely that *third differences involving graduated data beyond the original termini are to be all zero.* Differencing the added third differences three times more, we obtain

$$\Delta^6 u'_{n-2} = 0,\ \Delta^6 u'_{n-1} = 0, \ldots, \text{ and } \Delta^6 u'_{-3} = 0,\ \Delta^6 u'_{-4} = 0, \ldots,$$

from which, referring again to the difference equation, we must have

$$u'_{n+1} = u_{n+1},\ u'_{n+2} = u_{n+2} \ldots, \text{ and } u'_0 = u_0,\ u'_{-1} = u_{-1} \ldots \quad (12)$$

Thus the condition imposed on the auxiliary data is that graduated and ungraduated values are to coincide. It also follows that

$$\Delta^3 u_{n+1} = 0,\ \Delta^3 u_{n+2} = 0, \ldots, \text{ and } \Delta^3 u_{-3} = 0,\ \Delta^3 u_{-4} = 0, \ldots \quad (13)$$

The difference equation (2), with the conditions (3) and (12), or (3) and (13), suffices to determine the external data. If the original number of data to be graduated is not too small (in general, if it exceeds twenty-five), the external data may be found as accurately as may be required by the following method.

We have, from (12) and the difference equation (8),

$$u_x = -\frac{\epsilon E^3}{(E-1)^6 - \epsilon E^3} u_x,$$

or

$$\frac{(E-1)^6}{(E-1)^6 - \epsilon E^3} u_x = 0, \text{ for } x = n+1,\ n+2, \ldots.$$

This equation may be written

$$\frac{(\mathrm{E}-1)^3}{(\mathrm{E}-\alpha)(\mathrm{E}-\beta)(\mathrm{E}-\gamma)}\cdot\frac{(\mathrm{E}-1)^3}{(\mathrm{E}-\alpha^{-1})(\mathrm{E}-\beta^{-1})(\mathrm{E}-\gamma^{-1})}u_x=0. \qquad (14)$$

We shall show that $u_{n+1}, u_{n+2},\ldots$ are given by the equation formed by retaining only the first operating factor on the left side of (14), viz. the equation

$$\frac{(\mathrm{E}-1)^3}{(\mathrm{E}-\alpha)(\mathrm{E}-\beta)(\mathrm{E}-\gamma)}u_x=0. \qquad (15)$$

In the first place we see that (15) gives u_x uniquely in terms of $u_{x-1}, u_{x-2}\ldots$, for the operator can be expanded in *descending* powers of E, in the form

$$1-j_1\mathrm{E}^{-1}-j_2\mathrm{E}^{-2}-j_3\mathrm{E}^{-3}-\ldots,$$

so that we obtain

$$u_x=j_1u_{x-1}+j_2u_{x-2}+j_3u_{x-3}+\ldots, \qquad (16)$$

for $x = n+1, n+2, \ldots$. From this equation $u_{n+1}, u_{n+2}, \ldots$ are determined in succession, provided that the terms containing the undetermined external data $u_0, u_{-1}, \ldots$ are negligible.

Next, the values of u_x thus determined will satisfy (14); for the operator

$$\frac{(\mathrm{E}-1)^3}{(\mathrm{E}-\alpha^{-1})(\mathrm{E}-\beta^{-1})(\mathrm{E}-\gamma^{-1})}$$

is expansible in *ascending* powers of E, and therefore (14) can be deduced from (15), which holds for $x=n+1, n+2, \ldots$.

Lastly, the conditions (13) will be satisfied, for from (15) we have

$$\frac{(\mathrm{E}-1)^3}{(\mathrm{E}-\alpha)(\mathrm{E}-\beta)(\mathrm{E}-\gamma)}\cdot(\mathrm{E}-\alpha)(\mathrm{E}-\beta)(\mathrm{E}-\gamma)u_x=0,$$

since (15) holds for an ascending series of values of x.

Hence $\quad (\mathrm{E}-1)^3u_r=\Delta^3u_x=0$, for $x=n+1, n+2, \ldots$.

The value of j_n in (16) is easily found to be

$$j_n=\frac{-1}{r_1^2-2r_1r_2\cos\theta+r_2^2}\Delta^3{}_n\left\{r_1^{n-1}+\frac{r_2^{n-2}[r_2\sin(n-2)\theta-r_1\sin(n-1)\theta]}{\sin\theta}\right\}. \qquad (17)$$

By means of (17) the coefficients j_n have been calculated for the same values of ϵ as before, 0·01, 0·02, 0·05, 0·1, 0·25, and 1. It is found that they decrease successively with sufficient rapidity for all practical purposes.

Actually only three auxiliary data, u_{n+1}, u_{n+2}, u_{n+3}, need be calculated, for the rest can be found, again in virtue of (13), by forming a difference table with zero third differences. Exactly the same process may be applied to the other end of the table by reversing the order of the data.

We have finally to ascertain what values of ϵ are likely to be of standard practical application, and also to find a means, given any particular set of data to be graduated, of assigning in advance an appropriate value from among these. Here we may refer to the notion of a "smoothing coefficient", introduced by G. F. Hardy,[*] and based on the following considerations. The u's have been supposed subject to the same probable error, p, say. Then the probable error of their third differences, e.g. of $u_3 - 3u_2 + 3u_1 - u_0$, is $p\sqrt{(1+3^2+3^2+1)}$, or $p\sqrt{20}$. But if we have a graduating formula

$$u'_0 = \Sigma k_n u_n,$$

then

$$\Delta^3 u'_0 = \Sigma(\Delta^3 k_n) u_n,$$

so that the probable error of $\Delta^3 u'_0$ is $p\sqrt{\{\Sigma(\Delta^3 k_n)^2\}}$. The ratio $\sqrt{\{\Sigma(\Delta^3 k_n)^2\}} : \sqrt{20}$ may then be regarded as measuring the degree of reduction of probable error in third differences; it has been called the "smoothing coefficient" for the formula in question.

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[*] *J.I.A.* **32** (1896), p. 376. The name "smoothing coefficient" is due to J. Spencer. Cf. G. J. Lidstone, *J.I.A.* **42** (1908), p. 114.

TABLE OF COEFFICIENTS j_n

n.	$\epsilon=0{\cdot}01$.	$\epsilon=0{\cdot}02$.	$\epsilon=0{\cdot}05$.	$\epsilon=0{\cdot}1$.	$\epsilon=0{\cdot}25$.	$\epsilon=1$.
1	0·9155	1·0237	1·1847	1·3209	1·5204	1·8615
2	0·4491	0·4337	0·3812	0·3082	0·1523	−0·2545
3	0·1321	0·0626	−0·0600	−0·1750	−0·3468	−0·5842
4	−0·0563	−0·1293	−0·2297	−0·2965	−0·3471	−0·2660
5	−0·1442	−0·1917	−0·2309	−0·2294	−0·1700	0·0302
6	−0·1619	−0·1745	−0·1542	−0·1060	−0·0075	0·1257
7	−0·1374	−0·1199	−0·0633	−0·0027	0·0731	0·0899
8	−0·0937	−0·0579	0·0069	0·0531	0·0810	0·0276
9	−0·0474	−0·0065	0·0452	0·0654	0·0523	−0·0087
10	−0·0084	0·0268	0·0548	0·0512	0·0189	−0·0156
11	0·0185	0·0418	0·0460	0·0281	−0·0031	−0·0088
12	0·0330	0·0425	0·0293	0·0078	−0·0112	−0·0013
13	0·0369	0·0345	0·0127	−0·0045	−0·0100	0·0020
14	0·0334	0·0229	0·0005	−0·0091	−0·0053	0·0019
15	0·0257	0·0114	−0·0060	−0·0083	−0·0011	0·0008
16	0·0167	0·0024	−0·0079	−0·0052	0·0012	0·0000
17	0·0083	−0·0034	−0·0067	−0·0020	0·0017	−0·0003
18	0·0017	−0·0060	−0·0043	0·0003	0·0012	−0·0002
19	−0·0027	−0·0063	−0·0018	0·0013	0·0005	−0·0001
20	−0·0049	−0·0051	0·0000	0·0014	0·0000	...
21	−0·0055	−0·0034	0·0010	0·0010	−0·0002	
22	−0·0049	−0·0017	0·0012	0·0005	−0·0002	
23	−0·0037	−0·0003	0·0010	0·0001	−0·0001	
24	−0·0023	0·0006	0·0007	−0·0002	...	
25	−0·0010	0·0010	0·0003	−0·0002		
26	−0·0001	0·0010	0·0000	−0·0002		
27	0·0005	0·0008	−0·0001	−0·0001		
28	0·0008	0·0005	−0·0002	...		
29	0·0009	0·0003	−0·0002			
30	0·0007	0·0000	−0·0001			
31	0·0005	−0·0001	...			
32	0·0003	−0·0002				
33	0·0001	−0·0002				
34	0·0000	−0·0001				
35	−0·0001	−0·0001				
36	−0·0001	...				
37	−0·0001					
38	−0·0001					
39	−0·0001					
40	...					

Formula for extrapolating auxiliary data:

$$u_{n+r} = j_1 u_{n+r-1} + j_2 u_{n+r-2} + j_3 u_{n+r-3} + \dots \qquad (r > 0)$$

TABLE OF COEFFICIENTS k_n

n.	$\epsilon=0\cdot01$.	$\epsilon=0\cdot02$.	$\epsilon=0\cdot05$.	$\epsilon=0\cdot1$.	$\epsilon=0\cdot25$.	$\epsilon=1$.
0	0·1570	0·1769	0·2076	0·2347	0·2771	0·3601
1	0·1482	0·1644	0·1873	0·2056	0·2297	0·2604
2	0·1254	0·1329	0·1397	0·1412	0·1356	0·1045
3	0·0948	0·0928	0·0842	0·0721	0·0486	0·0023
4	0·0628	0·0536	0·0359	0·0191	−0·0046	−0·0293
5	0·0341	0·0216	0·0027	−0·0110	−0·0236	−0·0213
6	0·0117	−0·0004	−0·0145	−0·0211	−0·0211	−0·0056
7	−0·0035	−0·0125	−0·0191	−0·0186	−0·0108	0·0032
8	−0·0119	−0·0165	−0·0161	−0·0110	−0·0017	0·0044
9	−0·0148	−0·0151	−0·0099	−0·0035	0·0031	0·0023
10	−0·0138	−0·0110	−0·0038	0·0014	0·0039	0·0003
11	−0·0108	−0·0062	0·0005	0·0034	0·0027	−0·0006
12	−0·0070	−0·0021	0·0027	0·0033	0·0010	−0·0005
13	−0·0034	0·0008	0·0032	0·0022	−0·0001	−0·0002
14	−0·0005	0·0024	0·0026	0·0009	−0·0005	0·0000
15	0·0014	0·0028	0·0016	0·0000	−0·0005	0·0001
16	0·0023	0·0025	0·0006	−0·0004	−0·0003	0·0000
17	0·0025	0·0018	−0·0001	−0·0005	−0·0001	...
18	0·0022	0·0010	−0·0004	−0·0004	0·0001	
19	0·0017	0·0003	−0·0005	−0·0002	0·0001	
20	0·0010	−0·0001	−0·0004	0·0000	0·0000	
21	0·0005	−0·0004	−0·0002	0·0000	...	
22	0·0000	−0·0004	−0·0001	0·0001		
23	−0·0002	−0·0004	0·0000	0·0001		
24	−0·0004	−0·0003	0·0001	...		
25	−0·0004	−0·0002	0·0001			
26	−0·0003	0·0000	0·0001			
27	−0·0002	0·0000	...			
28	−0·0001	0·0001				
29	−0·0001	0·0001				
30	0·0000	0·0001				
31	0·0000	...				
32	0·0001					
33	0·0001					
34	0·0001					
35	...					

Formula of graduation:

$$u'_x = k_0 u_x + k_1(u_{x+1} + u_{x-1}) + k_2(u_{x+2} + u_{x-2}) + \dots.$$

Now each ϵ corresponds, as we have seen, to a definite linear combination, and therefore to a definite smoothing coefficient. Approximate values are given in the table below.

ϵ.	0	0·01	0·02	0·05	0·1	0·25	1	10	∞
Smoothing coefficient.	0	$\frac{1}{270}$	$\frac{1}{180}$	$\frac{1}{105}$	$\frac{1}{70}$	$\frac{1}{40}$	$\frac{1}{17}$	$\frac{1}{5}$	1

Thus a preliminary inspection of the third differences of a set of ungraduated data enables us to estimate the amount of smoothing likely to be required, and to select accordingly an appropriate value of ϵ.

155. **The Numerical Process of Graduation.**—The routine to be followed in applying the preceding theory to actual data falls therefore into three parts:

(1) To form the third differences of the given data, and by inspection to decide what degree of smoothing is necessary. To choose from the selected values of ϵ that which most nearly produces this degree of smoothing.

(2) To evaluate in succession four auxiliary data, u_{n+1}, u_{n+2}, u_{n+3}, u_{n+4} by the formula

$$u_{n+1}=j_1u_n+j_2u_{n-1}+j_3u_{n-2}+\ldots$$
$$u_{n+2}=j_1u_{n+1}+j_2u_n+j_3u_{n-1}+\ldots, \text{ \&c.}$$

As a check, the two second differences in the difference table of these four added data should be equal. Repeat these second differences outwards and build up the table to obtain as many auxiliary data as are required. Reverse the order of the data and carry out exactly the same process at the other end.

(3) To graduate the extended set by means of the coefficients k_n.

Ex.—To graduate, by this method, the set of data of § 144, p. 291.

We shall suppose that ϵ has been taken to be $0\cdot25$.

$$\begin{aligned}\text{Then } u_{46}&=1\cdot5204(1124)+0\cdot1523(1134)-0\cdot3468(1076)-\ldots=1155\cdot5,\\ u_{47}&=1\cdot5204(1155\cdot5)+0\cdot1523(1124)-0\cdot3468(1134)-\ldots=1156\cdot0,\\ u_{48}&=1\cdot5204(1156)+0\cdot1523(1155\cdot5)-0\cdot3468(1124)-\ldots=1142\cdot3,\\ u_{49}&=1\cdot5204(1142\cdot3)+0\cdot1523(1156)-0\cdot3468(1155\cdot5)-\ldots=1114\cdot4.\end{aligned}$$

Forming a difference table we find, as we should, that the two second differences are equal, each being $-14\cdot2$. By repeating them outwards and building up again we easily find the values* (taken to the nearest integer):

u_{50}	u_{51}	u_{52}	u_{53}	u_{54}	u_{55}	u_{56}	u_{57}	u_{58}
1072	1016	946	861	762	649	522	380	224

u_{59}	u_{60}	u_{61}	u_{62}	u_{63}	u_{64}
54	−130	−328	−541	−767	−1008

---

* No attempt should be made to *interpret* the auxiliary data in terms of the rest of the table. They are introduced purely to facilitate the solution.

Turning now to the other end of the table we obtain in the same manner

$$u_{19}=424\cdot3, \quad u_{18}=439\cdot0, \quad u_{17}=462\cdot8, \quad u_{16}=495\cdot7,$$

and by a difference table (second differences all 9·1)

u_{15}	u_{14}	u_{13}	u_{12}	u_{11}	u_{10}	u_9	u_8	u_7
538	589	649	718	797	884	981	1087	1201

u_6	u_5	u_4	u_3	u_2	u_1
1325	1458	1600	1752	1912	2081

The final process of graduating the extended table, and indeed all processes which, like the method of Least Squares and the present method, depend on forming a symmetrical linear compound, but are not reducible to successive "summation", are most conveniently carried out when arranged as on table facing p. 316. On a sheet of computing paper the data to be graduated are entered in a column at equal distances below each other, and to the left other columns of *strictly equal width* are allotted to each successive graduating coefficient, which is written at the head of its column. The products $k_m u_n$ in each column are computed either by arithmometer or, in three place work, by Crelle's Tables. The graduated value u' of any u is then seen to be the sum of the entries in the diagonal lines which converge in the k_0 column opposite to that u.

Part of the whole computing sheet is shown facing p. 316. In summing the diagonal entries, distraction of the eye may be avoided by the use of a V-shaped stencil, which leaves exposed at any time only the particular converging diagonals required.

Checks during the computation may be provided by summing the entries in any completed row, the entry in the column under k_0 being halved. In the row opposite u_m the sum should be $\frac{1}{2}u_m$. A final check is provided by the "Theorems of Conservation" of § 153. Thus in the present example we find the three moments to be

18285, 290109, 5581363 for the ungraduated data,
18284, 290095, 5581071 for the graduated data.

A comparison of the graduated and ungraduated data and their third differences shows that a satisfactory degree of smoothing has been secured without great departure from the original values.

156. Other Methods.—In addition to the methods of graduation which have been described in this chapter, mention should be made of a method proposed by E. C. Rhodes, and described in his Tract* on *Smoothing*, to which the reader is referred. A memoir by C. Lanczos, "Trigonometric interpolation of empirical and analytical functions", *Journ. Math. Phys.* **17** (1938) 123, may also be consulted with advantage.

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* No. VI of the *Tracts for Computers*, edited by K. Pearson; Camb. Univ. Press (1921).

1.	1.	1856.	2297.	2771.	x.	u_x		w_x	
						...			
1	1					718			
1	1					649			
1	1					589			
1	1					538			
0	0					496			
0	0					463			
0	0	595				439	Δ^3		Δ^3
0	0	575	974			424			
0	0	584	990	1194	20	431	− 69	419	−4
0	0	555	939	1133	1	409	142	422	−6
0	0	582	985	1189	2	429	−248	435	−5
0	0	572	969	1169	3	422	112	454	0
1	1	719	1217	1469	4	530	107	473	8
1	1	685	1160	1399	25	505	− 99	487	3
0	0	622	1054	1272	6	459	23	496	0
0	0	677	1146	1383	7	499	− 23	508	−2
1	1	713	1208	1458	8	526	− 3	526	0
1	1	763	1293	1560	9	563	60	550	0
1	1	796	1348	1627	30	587	− 74	578	−1
1	1	808	1367	1649	1	595	85	610	−3
1	1	877	1486	1793	2	647	−118	646	−5
1	1	907	1537	1854	3	669	67	685	0
1	1	1012	1714	2067	4	746	28	724	1
1	1	1031	1746	2106	35	760	− 64	758	4
1	1	1055	1787	2156	6	778	4	787	6
1	1	1123	1902	2294	7	828	118	812	5
1	1	1147	1943	2344	8	846	−130	837	−2
1	1	1134	1920	2317	9	836	58	868	−5
1	1	1242	2104	2538	40	916	− 14	910	−6
1	1	1296	2196	2649	1	956	− 8	961	−8
1	1	1375	2329	2810	2	1014	− 64	1016	−4
1	1	1459	2472	2982	3	1076		1069	
1	1	1538	2605	3142	4	1134		1112	
1	1	1524	2582	3115	45	1124		1141	
1	1	1568	2655			1156			
1	1	1568				1156			
1	1					1142			
1	1					1114			
1	1					1072			
1	1					1016			
1	1					946			
1	1					861			
						...			

CHAPTER XII

CORRELATION

157. Definition of Correlation.—Consider a definite group containing a large number of individuals; let us measure some attribute A of the individuals, and let us also measure some other attribute B. For instance, the individuals might be all the stars of the third magnitude, and A might represent the parallax of the star, while B might represent its proper motion; or the group might consist of all adult Scotsmen, and A might represent the height of a man in inches, while B might represent his wealth in pounds sterling. Consider now the individuals in the group for whom A lies between x and $x+dx$ while B lies between y and $y+dy$; let the number of such individuals be $N\phi(x, y)dxdy$, where N denotes the total number of individuals in the group, or, to express the same thing in other words, let $\phi(x, y)dxdy$ denote the *probability* that for an individual taken at random the first attribute A lies between x and $x+dx$, while the second attribute lies between y and $y+dy$.

Now Fermat's Principle of Conjunctive Probability may be stated thus: *The probability that two events will both happen is* hk, *where* h *is the probability that the first event will happen, and* k *is the probability that the second event will happen when the first event is known to have happened.* Applying this to the present case, let $h=f(x)dx$ be the probability that for an individual taken at random from the group the first attribute A lies between x and $x+dx$; and let $k=g(x, y)dy$ be the probability that for an individual taken at random from those members of the group whose attribute A lies between x and

$x + dx$, the attribute B lies between y and $y + dy$. Then by the Principle of Conjunctive Probability we have

$$\phi(x, y) = f(x)g(x, y).$$

Now here two possibilities present themselves.

In the first possibility, $g(x, y)$ is a function of y only, not involving x. When this is the case, if we divide the original group into sub-groups according to the magnitude of the attribute A, then the probability that B will lie between y and $y + dy$ is *the same for each of the sub-groups.* The two attributes A and B are then said to be *not correlated,* and evidently $\phi(x, y)$ *is expressible as the product of a function of x only, multiplied by a function of y only.* This would be the case, approximately at least, with the height and wealth of the Scotsmen; for let the probability that a man is of a certain height x to $x + dx$ be $f(x)dx$; then the probability that his wealth lies between y and $y + dy$ pounds is nearly the same for tall men as for short men, so may be expressed in the form $g(y)dy$, where $g(y)$ does not involve x; and the compound probability that his height is between x and $x + dx$ while his wealth is between y and $y + dy$ is then simply

$$f(x)g(y)dxdy.$$

But in a large class of cases the function $\phi(x, y)$ is not capable of being expressed as a function of x multiplied by a function of y. In such cases the probability that the first attribute has a measure between x and $x + dx$ is not the same for individuals with large y's as for individuals with small y's, and the probability that the second attribute has a measure between y and $y + dy$ is not the same for individuals with large x's as for individuals with small x's. In such cases the two attributes are said to be *correlated.* Thus the parallaxes and the proper motions of the stars are correlated; for a star which has a large parallax, and is therefore comparatively near to us, is more likely to have a large proper motion than a small one.

Many elementary problems in Probability cannot be solved correctly without taking account of correlation. For example, the following: "The probability that A can solve a mathematical problem taken at random from a certain book is $\frac{1}{2}$, and the probability that B can solve one is also $\frac{1}{2}$. What is the probability that a problem taken at random will be solved by one or other or both of them?"

Here the group consists of all the problems in the book, and the two attributes of an individual problem are its solubility by A and its solubility by B. These two attributes are correlated, since the problems that A can solve will be, to a great extent, the same as the problems that B can solve. It would therefore be wrong to assert that the probability of both failing to solve a problem taken at random is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and that consequently the probability that one or other or both would succeed is $1 - \frac{1}{4}$ or $\frac{3}{4}$.

158. **An Example of a Frequency Distribution involving Correlation.**—As an illustration of correlation, let us consider two riflemen side by side firing at targets when a strong wind is blowing. The wind will be supposed to affect the shooting of both men in much the same way, so that we may expect a certain amount of correlation between their records. In this case the "group" consists of all the records of the two men's shots, an "individual" of the group is constituted of a single shot of the first man together with the shot fired at the same instant by the second man, and the "attributes" of this individual are the deviations of the two shots.

Let X denote that part of the deviation of the first man's bullet from the mark which is due to causes affecting him alone and not affecting the other man, *i.e.* all causes except the wind. Similarly let Y denote that part of the second man's deviation which is due to causes affecting him only; and let $X + aZ$ denote the total deviation of the first man's bullet, and $Y + bZ$ denote the total deviation of the second man's bullet, where Z is due to the wind. For simplicity, we suppose all the deviations to be in a horizontal direction from the mark. We assume that X, Y, Z are independent of each other, and that each occurs according to the normal law of frequency, so the probability that

X lies between x and $x + dx$ is $\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx$,

Y " " y and $y + dy$ is $\frac{k}{\sqrt{\pi}} e^{-k^2y^2} dy$,

Z " " z and $z + dz$ is $\frac{l}{\sqrt{\pi}} e^{-l^2z^2} dz$.

We now want to find the frequency of cases in which $U = X + aZ$ has a value between u and $u + du$, and $V = Y + bZ$ has a value between v and $v + dv$.

Suppose Z lies between z and $z + dz$, which happens in the proportion $\frac{l}{\sqrt{\pi}} e^{-l^2z^2} dz$ of cases. Then in order to produce such a pair as is considered, X must be taken between $u - az$ and $u + du - az$, while Y must be between $v - bz$ and $v + dv - bz$; the probability of these happening together is

$$\frac{h}{\sqrt{\pi}} e^{-h^2(u-az)^2} du \times \frac{k}{\sqrt{\pi}} e^{-k^2(v-bz)^2} dv.$$

Therefore the frequency of cases in which U lies between u and $u + du$, while V lies between v and $v + dv$, is

$$\frac{hkl}{\pi^{\frac{3}{2}}} du dv \int_{-\infty}^{\infty} e^{-l^2z^2 - h^2(u-az)^2 - k^2(v-bz)^2} dz,$$

or

$$\frac{hkl}{\pi^{\frac{3}{2}}} du dv e^{-u^2h^2 - v^2k^2 + \frac{(auh^2+bvk^2)^2}{a^2h^2+b^2k^2+l^2}} \int_{-\infty}^{\infty} e^{-(a^2h^2+b^2k^2+l^2)\left\{z - \frac{auh^2+bvk^2}{a^2h^2+b^2k^2+l^2}\right\}^2} dz,$$

or
$$\frac{1}{\pi} \frac{hkl}{(a^2h^2+b^2k^2+l^2)^{\frac{1}{2}}} e^{-u^2h^2 - v^2k^2 + \frac{(auh^2+bvk^2)^2}{a^2h^2+b^2k^2+l^2}} du dv.$$

If we examine this expression we see that it is of the form

$$\phi(u, v) du dv \equiv c e^{-p^2u^2 - q^2v^2 + 2suv} du dv, \qquad (1)$$

where c, p, q, s denote constants. The constant s would be zero if either of the constants a or b were zero, *i.e.* if no common influence acted on the two riflemen. *The correlation is represented analytically by the occurrence of this term in* uv *in the exponential.* If this term were absent, the expression (1) could be regarded as the product of two factors

$$\text{Constant} \times e^{-p^2u^2} du \text{ and } \text{Constant} \times e^{-q^2v^2} dv,$$

of which the first involves u only and the second involves v only, so that in this case (§ 157) there would be no correlation.

The expression (1) may be regarded as the extension to two variables u and v of the normal law of frequency for one variable u,

$$\frac{h}{\sqrt{\pi}} e^{-h^2u^2} du;$$

on this account we shall call it the *normal law of frequency for two variables.* The constant c in the expression (1) may be determined from the condition that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u, v)dudv = 1$; this gives

$$c = \frac{1}{\pi}(p^2q^2 - s^2)^{\frac{1}{2}}. \tag{2}$$

The above is a particular case of the following more general result.*

Let $u_1, u_2, \ldots, u_\sigma$ be variables, such that the probability of u_i lying between v_i and $v_i + dv_i$ is

$$\frac{h_i}{\sqrt{\pi}}e^{-h_i^2v_i^2}dv_i.$$

Let $x_1, x_2, \ldots, x_\rho$ be a set of linear functions of the u's, defined by equations

$$\begin{cases} x_1 = k_{11}u_1 + k_{12}u_2 + \ldots + k_{1\sigma}u_\sigma, \\ x_2 = k_{21}u_1 + k_{22}u_2 + \ldots + k_{2\sigma}u_\sigma, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_\rho = k_{\rho 1}u_1 + k_{\rho 2}u_2 + \ldots + k_{\rho\sigma}u_\sigma, \end{cases}$$

and let

$$a_{ji} = \frac{k_{ji}}{h_i}.$$

Then the probability that x_j lies between ξ_j and $\xi_j + d\xi_j$ is

$$\sqrt{\left(\frac{\mathrm{E}}{\pi\rho}\right)} \cdot e^{-(\sum_1^\rho b_{jj}\xi_j^2 + 2\sum_1^\rho b_{jk}\xi_j\xi_k)}d\xi_1 d\xi_2 \ldots d\xi_\rho,$$

where $b_{jk} = (-1)^{j+k}\frac{\Sigma \mathrm{D}_j\mathrm{D}_k}{\Sigma \mathrm{D}^2}$ $(j, k = 1, 2, \ldots \rho)$ and $\mathrm{E} = \frac{1}{\Sigma \mathrm{D}^2}$,

while D denotes a determinant of order ρ taken from the array

$$\begin{matrix} a_{11}a_{12} & \ldots & a_{1\sigma}, \\ a_{21}a_{22} & \ldots & a_{2\sigma}, \\ \cdot & \cdot & \cdot \\ a_{\rho 1}a_{\rho 2} & \ldots & a_{\rho\sigma}, \end{matrix}$$

and D_j denotes a determinant of order $(\rho - 1)$ taken from the array which is obtained by omitting the jth row in the above array. The two determinants D_j and D_k in the product $\mathrm{D}_j\mathrm{D}_k$ are to be formed from the same columns of M.

159. **Bertrand's Proof of the Normal Law.**—It was remarked by Bertrand † that the normal law of frequency for two variables may be deduced from an assumption resembling the Postulate of the Arithmetic Mean, from which, as we have seen

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* Cf. M. J. van Uven, *Proc. Amster. Ac.* **16** (1914), p. 1124.

† *Comptes Rendus*, **106** (1888), p. 387.

(§ 112), the normal law of frequency for one variable may be deduced.

Consider for definiteness the shots of a rifleman at a target; and let the horizontal and vertical deviations of a shot from the centre of the target be called A and B. It is found in practice that A and B are not independent, but are to some extent correlated.

Let us now make the following assumption, which was first proposed by Cotes: that *if any number of shots have struck the target in points* $P_1, P_2, \ldots, P_n$, *then the most probable position of the point aimed at is the centroid* (*centre of gravity*) *of these points.*

Suppose the probability that a shot strikes an element of area $dxdy$ at (x, y) is

$$F(X-x, Y-y)dxdy,$$

when X, Y are the co-ordinates of the point aimed at. Then if the co-ordinates of the points $P_1, P_2, \ldots, P_n$ are (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ respectively, the product

$$F(X-x_1, Y-y_1)F(X-x_2, Y-y_2) \ldots F(X-x_n, Y-y_n)$$

must be a maximum, when X and Y regarded as variables have for values

$$X=\frac{x_1+x_2+\ldots+x_n}{n},$$

$$Y=\frac{y_1+y_2+\ldots+y_n}{n}.$$

If then we put

$$\frac{\partial \log F(x, y)}{\partial x}=\phi(x, y), \qquad \frac{\partial \log F(x, y)}{\partial y}=\psi(x, y),$$

$$X-x_n=a_n, \qquad Y-y_n=\beta_n,$$

the functions ϕ and ψ must be such that the equations

$$\left.\begin{array}{l}\phi(a_1, \beta_1)+\phi(a_2, \beta_2)+\ldots+\phi(a_n, \beta_n)=0\\ \psi(a_1, \beta_1)+\psi(a_2, \beta_2)+\ldots+\psi(a_n, \beta_n)=0\end{array}\right\} \qquad (1)$$

are the necessary consequences of

$$\left.\begin{array}{l}a_1+\ldots+a_n=0\\ \beta_1+\ldots+\beta_n=0\end{array}\right\}. \qquad (2)$$

Therefore

$$\phi(-\alpha_2-\alpha_3-\ldots-\alpha_n,\ -\beta_2-\beta_3-\ldots-\beta_n) \\ +\phi(\alpha_2,\beta_2)+\ldots+\phi(\alpha_n,\beta_n)$$

is identically zero; and therefore differentiating partially with respect to α_2 and denoting the partial derivatives of $\phi(x, y)$ with respect to x and y by ϕ_1 and ϕ_2 respectively, we have

$$-\phi_1(-\alpha_2-\ldots-\alpha_n,\ -\beta_2-\ldots-\beta_n)+\phi_1(\alpha_2,\beta_2)=0,$$

so $\phi_1(x, y)$ is independent of x and y: denote it by a. Similarly $\phi_2(x, y)$ is a constant b, and so

$$\phi(x, y)=ax+by+c.$$

Since equations (1) are consequences of equation (2), we see on substituting this value of ϕ that c is zero. Thus

$$\phi(x, y)=ax+by.$$

Similarly $$\psi(x, y)=a'x+b'y,$$

where a' and b' are constants.

Since ϕ and ψ are partial derivatives of the same function, we must have $b=a'$, and thus, integrating,

$$\log \mathrm{F}(x, y)=\tfrac{1}{2}ax^2+bxy+\tfrac{1}{2}b'y^2+\text{constant}.$$

Therefore $\mathrm{F}(x, y)$ is of the form

$$\text{Constant}\times e^{\frac{1}{2}ax^2+bxy+\frac{1}{2}b'y^2},$$

and *the normal law of frequency for two variables is thus established.*

The study of normal frequency distributions in two and three variables was begun by August Bravais in a celebrated memoir, *Sur les probabilités des erreurs de situation d'un point*, published in 1846.*

Ex.—The displacement of a point is the vector sum of n displacements, and the probability that the i^{th} of these displacements has a value between (x_i, y_i) *and* (x_i+dx_i, y_i+dy_i) *is*

$$\frac{\delta_i^{\frac{1}{2}}}{\pi}e^{-(\alpha_i x_i^2+2\beta_i x_i y_i+\gamma_i y_i^2)}dx_i dy_i,$$

where $\delta_i=\alpha_i\gamma_i-\beta_i^2$ $(i=1,\ 2,\ \ldots,\ n)$. *Show that the probability that the total displacement has a value between* (x, y) *and* $(x+dx, y+dy)$ *is*

$$\frac{\delta^{\frac{1}{2}}}{\pi}e^{-(\alpha x^2+2\beta xy+\gamma y^2)}dxdy,$$

---

[*] *Mém. Sav. Étrang.*, Paris, 9 (1846), p. 255.

where $\alpha=\frac{A}{\Delta}$, $\beta=\frac{B}{\Delta}$, $\gamma=\frac{C}{\Delta}$, $\delta=\frac{1}{\Delta}$, *the numbers* A, B, C, Δ *being defined by the equations*

$$A=\sum_{i=1}^{n}\frac{\alpha_i}{\delta_i},\quad B=\sum_{i=1}^{n}\frac{\beta_i}{\delta_i},\quad C=\sum_{i=1}^{n}\frac{\gamma_i}{\delta_i},\quad \Delta=AC-B^2.$$

(d'Ocagne.)

160. The More General Law of Frequency.—In § 86 we saw that the normal law of frequency for one variable was a special, though frequently occurring, case of a more general law of frequency. Denoting the probability of a deviation between x and $x+dx$ by $\phi(x)dx$, then in this more general law $\phi(x)$ is represented by an infinite series, whose first term is $\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$, and whose subsequent terms are obtained from this term by differentiation. The normal law corresponds to the case when the infinite series reduces to its first term.

Similarly in the case when there are two variables we may derive a more general law of frequency than the normal law, represented by $\phi(x, y)dxdy$, where

$$\phi(x, y)=c_0f(x, y)+c_{10}\frac{\partial f}{\partial x}+c_{01}\frac{\partial f}{\partial y}+c_{20}\frac{\partial^2 f}{\partial x^2}+c_{11}\frac{\partial^2 f}{\partial x\partial y}+c_{02}\frac{\partial^2 f}{\partial y^2}+\dots$$

where $f(x, y)=e^{-p^2x^2-q^2y^2+2sxy}$, when the origin is suitably chosen, or more generally

$$f(x, y)=e^{-p^2(x-m)^2-q^2(y-n)^2+2s(x-m)(y-n)},$$

where p, q, s, m, n are constants.

161. Determination of the Constants in a Normal Frequency Distribution with Two Variables.—Let h be the smallest step recognised in measuring x, and let k be the smallest step in measuring y; and let the probability that the first attribute A has a measure between x and $x+h$, while the second attribute B has a measure between y and $y+k$, be $\phi(x,y)hk$, where (§ 158)

$$\phi(x,y)=\frac{1}{\pi}(p^2q^2-s^2)^{\frac{1}{2}}e^{-p^2(x-a)^2+2s(x-a)(y-b)-q^2(y-b)^2}.$$

Let it be required to determine the most probable values of the constants p, q, s, a, b, from a set of observations. Let the measures of the individuals observed be (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n). Then the *a priori* probability that the observations will yield these measures is

$$\frac{h^nk^n}{\pi^n}(p^2q^2-s^2)^{\frac{1}{2}n}e^{-p^2\Sigma(x_1-a)^2+2s\Sigma(x_1-a)(y_1-b)-q^2\Sigma(y_1-b)^2}$$

The most probable hypothesis regarding p, q, s, a, b is that which makes this quantity a maximum when $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are supposed given. Taking logs, we see that

$$\Pi \equiv \tfrac{1}{2}n \log (p^2q^2 - s^2) - p^2\Sigma(x_1 - a)^2 + 2s\Sigma(x_1 - a)(y_1 - b) - q^2\Sigma(y_1 - b)^2$$

must be a maximum, and therefore

$$\frac{\partial\Pi}{\partial a} = 0, \quad \frac{\partial\Pi}{\partial b} = 0, \quad \frac{\partial\Pi}{\partial p} = 0, \quad \frac{\partial\Pi}{\partial q} = 0, \quad \frac{\partial\Pi}{\partial s} = 0.$$

The first two of these equations are

$$0 = p^2\Sigma(x_1 - a) - s\Sigma(y_1 - b),$$
$$0 = -s\Sigma(x_1 - a) + q^2\Sigma(y_1 - b),$$

which, since $pq \neq s$, give at once

$$a = \frac{1}{n}\Sigma x_1$$

and

$$b = \frac{1}{n}\Sigma y_1.$$

The other three equations are

$$0 = \frac{npq^2}{p^2q^2 - s^2} - 2p\Sigma(x_1 - a)^2,$$
$$0 = \frac{np^2q}{p^2q^2 - s^2} - 2q\Sigma(y_1 - b)^2,$$
$$0 = -\frac{ns}{p^2q^2 - s^2} + 2\Sigma(x_1 - a)(y_1 - b).$$

Let us denote $\frac{1}{n}\Sigma(x_1 - a)^2$ by σ_1^2, $\frac{1}{n}\Sigma(y_1 - b)^2$ by σ_2^2, and $\frac{1}{n\sigma_1\sigma_2}\Sigma(x_1 - a)(y_1 - b)$ by r, so that the three numbers σ_1, σ_2, r may be calculated from the observed measures. Then the three preceding equations may be written

$$\frac{p^2}{2\sigma_2^2} = \frac{q^2}{2\sigma_1^2} = \frac{s}{2\sigma_1\sigma_2 r} = p^2q^2 - s^2.$$

Each fraction is evidently equal to $\frac{1}{2\sigma_1\sigma_2}\sqrt{\left(\frac{p^2q^2 - s^2}{1 - r^2}\right)}$, and therefore

$$p^2q^2 - s^2 = \frac{1}{4(1 - r^2)\sigma_1^2\sigma_2^2}.$$

Thus the values of p, q, r are given by

$$p^2 = \frac{1}{2(1-r^2)\sigma_1^2}, \quad s = \frac{r}{2(1-r^2)\sigma_1\sigma_2}, \quad q^2 = \frac{1}{2(1-r^2)\sigma_2^2}.$$

and replacing p, q, s by these values in the frequency function we see that *the probability that the first attribute has a measure between x and $x+h$, while the second attribute has a measure between y and $y+k$, is $\phi(x, y)hk$, where*

$$\phi(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}} e^{-\frac{1}{2(1-r^2)}\left\{\frac{(x-a)^2}{\sigma_1^2} - \frac{2r(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2}\right\}}, \quad (1)$$

and where the constants a, b, σ_1, σ_2, r are expressed in terms of the observed measures by the equations

$$a = \frac{1}{n}\Sigma x_1,$$

$$b = \frac{1}{n}\Sigma y_1,$$

$$\sigma_1^2 = \frac{1}{n}\Sigma(x_1 - a)^2,$$

$$\sigma_2^2 = \frac{1}{n}\Sigma(y_1 - b)^2,$$

$$r = \frac{1}{n\sigma_1\sigma_2}\Sigma(x_1 - a)(y_1 - b).$$

These formulae enable us to determine the most probable values of the constants of a normal frequency distribution in two variables in terms of observed measures.

Ex.—*A man is firing at a target, aiming at its centre. Taking this centre as origin, the co-ordinates of the points struck by the bullets in n shots are (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n). Writing*

$$\frac{1}{n}(x_1^2 + x_2^2 + \ldots + x_n^2) = \sigma_1^2,$$

$$\frac{1}{n}(y_1^2 + y_2^2 + \ldots + y_n^2) = \sigma_2^2,$$

$$\frac{1}{n}(x_1y_1 + x_2y_2 + \ldots + x_ny_n) = \sigma_1\sigma_2 r,$$

show that in the long run one-half of the points struck will lie within the ellipse whose equation is

$$\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)=0{\cdot}69315.$$

(Bertrand, *C.R.* **106** (1888), p. 521.)

162. **The Frequencies of the Variables taken Singly.**—Let us now find the probability that the attribute A lies between x and $x+dx$, when the attribute B is ignored, in the normal frequency distribution which has been studied. By (1) of the last article, the required probability is $n_x dx$, where

$$n_x=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}}\int_{-\infty}^{\infty}e^{-\frac{1}{2(1-r^2)}\left\{\frac{(x-a)^2}{\sigma_1^2}-\frac{2r(x-a)(y-b)}{\sigma_1\sigma_2}+\frac{(y-b)^2}{\sigma_2^2}\right\}}dy.$$

Performing the integration, remembering that

$$\int_{-\infty}^{\infty}e^{-(p+2sz+qz^2)}dz=\frac{\sqrt{\pi}}{\sqrt{q}}e^{-p+\frac{s^2}{q}},$$

we have
$$n_x=\frac{1}{\sigma_1\sqrt{(2\pi)}}e^{-\frac{(x-a)^2}{2\sigma_1^2}}.$$

This equation shows that σ_1 *is the standard deviation for the attribute* A, when the attribute B is ignored. Similarly σ_2 is the standard deviation for the attribute B.

Denoting by $\phi(x, y)dxdy$ the probability that the attribute A lies between x and $x+dx$ while the attribute B lies between y and $y+dy$, and denoting by $g(x, y)dy$ the probability that B lies between y and $y+dy$ when A is known to be between x and $x+dx$, we have

$$\phi(x, y)=n_x g(x, y).$$

Substituting in this equation the known values of ϕ and n_x, we obtain $g(x, y)$, and thus find that *the probability that* B *lies between y and $y+dy$, when* A *is known to be between x and $x+dx$, is*

$$\frac{1}{\sigma_2\sqrt{\{2\pi(1-r^2)\}}}e^{-\frac{1}{2(1-r^2)\sigma_2^2}\left\{y-b-\frac{\sigma_2 r(x-a)}{\sigma_1}\right\}^2}dy.$$

This is a normal frequency distribution about the mean $b+\frac{\sigma_2 r(x-a)}{\sigma_1}$, *with the standard deviation* $\sigma_2\sqrt{(1-r^2)}$.

It follows from this that if (with a great number of observations) we find the mean y_M of all the measured values of B for which the measured value of A lies between x and $x + dx$, and *if then we plot* y_M *against* x, *the plotted points will lie on the straight line*

$$y_M - b = \frac{\sigma_2 r}{\sigma_1}(x - a).$$

Similarly if x_M denotes the mean of all the measured values of A for which the measured value of B lies between y and $y + dy$, then x_M plotted against y gives the straight line

$$y - b = \frac{\sigma_2}{\sigma_1 r}(x_M - a).$$

These lines were called by Galton *lines of regression.*

Ex. 1.—*To find the standard deviation of the difference between the measures of the two attributes* A *and* B.

The probability that (measure of A) − (measure of B) lies between $a - b + x$ and $a - b + x + dx$ is evidently

$$\frac{dx}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}}\int_{-\infty}^{\infty} dv \,.\, e^{-\frac{(v+x)^2}{2\sigma_1^2(1-r^2)} - \frac{v^2}{2\sigma_2^2(1-r^2)} + \frac{2rv(v+x)}{2\sigma_1\sigma_2(1-r^2)}},$$

or

$$\frac{dx}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}} e^{\frac{-x^2}{2(\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2)}}\int_{-\infty}^{\infty} dv \,.\, e^{-\frac{\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2}{2\sigma_1^2\sigma_2^2(1-r^2)}\left(v - \frac{-x\sigma_2^2+rx\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2}\right)^2},$$

or

$$\frac{dx}{\sqrt{(2\pi)}\sqrt{(\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2)}} e^{-\frac{x^2}{2(\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2)}}.$$

Therefore if σ_ν denote the standard deviation of $(x - y)$, we have

$$\sigma_\nu^2 = \sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2,$$

and therefore
$$r = \frac{\sigma_1^2 + \sigma_2^2 - \sigma_\nu^2}{2\sigma_1\sigma_2},$$

an equation which may be used to determine r.*

Ex. 2.†—*Show that the standard deviation of the sum of the measures of* A *and* B *is*

$$(\sigma_1^2 + \sigma_2^2 + 2r\sigma_1\sigma_2)^{\frac{1}{2}}.$$

Ex. 3.—*Show that the standard deviation of the product of the measures of* A *and* B *is*

$$\{b^2\sigma_1^2 + a^2\sigma_2^2 + 2rab\sigma_1\sigma_2 + \sigma_1^2\sigma_2^2(1 + r)\}^{\frac{1}{2}}.$$

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* Cf. K. Pearson, *Drapers' Company Research Memoirs*, Biometric Series, IV. (1907).

† Cf. R. Pearl, *Biometrika*, **6** (1909), p. 437.

163. **The Coefficient of Correlation.**—We now approach the question, How is correlation to be measured?

We have seen that in a normal frequency distribution for two variables defined by the frequency function

$$\phi(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}} e^{-\frac{1}{2(1-r^2)}\left\{\frac{(x-a)^2}{\sigma_1^2} - \frac{2r(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2}\right\}}$$

the existence of correlation depends on the presence of the term in $(x-a)(y-b)$ in the exponential, *i.e.* it depends on the coefficient r. When r is zero there is no correlation, since $\phi(x, y)$ then factorises into the product of a term depending on x only and a term depending on y only.

Consider the case when there is *perfect* correlation, *i.e.* each value of the measure of A occurs only in conjunction with a particular value of the measure of B, so that one determines the other. In this case the standard deviation of B, when A is known to have a definite value, must be zero; that is, by the last section, $\sigma_2\sqrt{(1-r^2)}$ must vanish, and therefore r must have the value unity.

Thus $r=0$ corresponds to the absence of correlation, while $r=1$ corresponds to perfect correlation. It is therefore natural to take either r itself or some power of it, such as r^2 or $\sqrt{r}$, as the numerical measure of the correlation between the attributes A and B. To decide which power is most suitable,* let us recur to the case of the two riflemen. If we suppose, in the notation of § 158, that a and b are each unity while h, k, l are equal to each other, the frequency function becomes

$$\frac{h^2}{\pi\sqrt{3}} e^{-\frac{2h^2u^2}{3} + \frac{2h^2uv}{3} - \frac{2h^2v^2}{3}},$$

which corresponds to $\sigma_1^2 = \sigma_2^2 = \frac{1}{h^2}$, $r = \frac{1}{2}$. But in this case exactly half of each man's mean error is due to the common element (the wind), and it would seem natural to take the measure of correlation to be $\frac{1}{2}$. We therefore decide that r *itself is the most suitable numerical measure of correlation.* It is called the *coefficient of correlation.*

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* Kapteyn, *Monthly Notices R.A.S.* **72** (1912), p. 518.

Let (x_1, y_1), (x_2, y_2), . . . be a great number of measures of the attributes A and B; and let points having these co-ordinates be plotted. The points will cluster round the mean $x=a$, $y=b$; and if we draw rectangular axes through this mean point, the majority of the points will be found in the first and third of the quadrants formed by these axes when r is positive, and in the second and fourth quadrants when r is negative.

164. Alternative Way of computing the Correlation Coefficient.—We have seen in § 97 that the standard deviation of a normal frequency distribution in a single variable may be found in many different ways from given observational material; the same applies to the coefficient of correlation in a normal frequency distribution in two variables. Thus to obtain the correlation coefficient directly from the raw material without attempting to arrange either of the measurements in order of increasing magnitude, we may proceed as follows: First summing each column, find the mean a of the x's and the mean b of the y's. Take out all the x's that exceed a, find their mean A, and find also the mean B′ of the corresponding y's. Then

$$\sigma_1=\sqrt{\frac{\pi}{2}}(\mathrm{A}-a),\quad \text{and } \mathrm{B}'-b=\frac{r\sigma_2}{\sigma_1}(\mathrm{A}-a)\quad \text{or } \mathrm{B}'-b=\sqrt{\frac{2}{\pi}}\,.\,r\sigma_2.$$

Similarly take out all the y's that exceed b, find their mean B, and find also the mean A′ of the corresponding x's. We have

$$\sigma_2=\sqrt{\frac{\pi}{2}}(\mathrm{B}-b),\quad \mathrm{A}'-a=\frac{r\sigma_1}{\sigma_2}(\mathrm{B}-b)\quad \text{or } \mathrm{A}'-a=\sqrt{\frac{2}{\pi}}\,.\,r\sigma_1.$$

These equations give σ_1, σ_2, and r. In fact

$$r=\frac{\mathrm{B}'-b}{\mathrm{B}-b}=\frac{\mathrm{A}'-a}{\mathrm{A}-a}.$$

165. Numerical Examples.—

Ex. 1.—As a first example we shall consider the results of throws of dice made by A. D. Darbishire.*

Twelve dice were taken, of which m were marked with red, the rest being white. All 12 dice were thrown together, and the number of dice showing faces with 4 or more pips uppermost in this throw was noted; this number will be called the "First Throw."

The red dice were left down and the white dice thrown again. The total number of dice (red and white) now showing faces with 4 or more pips uppermost was noted; this will be called the "Second Throw" corresponding to the "First Throw" previously made.† Evidently there will be correlation between the first and second throws.‡

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* *Mem. Manchester Lit. and Phil. Soc.* **51** (1907), No. 16.

† It may be shown without difficulty that the probability of the case in which the first throw is p and the corresponding second throw is q is the coefficient of $y^p z^q$ in the expansion of

$$(\tfrac{1}{2})^{24-m}(1+yz)^m(1+y)^{12-m}(1+z)^{12-m}.$$

‡ In fact, $r=\frac{m}{12}$.

For 500 pairs of throws of 12 dice, of which 6 were marked red and were left down and counted again in the second throw, the results were as in the following table, a pair of throws of (say) 2 and 5 being entered as a unit in the square at the intersection of the third row and sixth column.

SECOND THROWS

FIRST THROWS	0	1	2	3	4	5	6	7	8	9	10	11	12	Totals
0														0
1			1	1	1									3
2			1		2	3	2							8
3			2	3	5	6	2	6						24
4			5	9	8	11	16	7	6	1				63
5			2	5	17	24	19	25	11	2				105
6			1	5	14	25	24	24	17	4	3			117
7				2	2	13	16	27	12	4	2			78
8					2	7	13	22	14	5	3			66
9						3	5	6	9	5	2			30
10								2	1	2				5
11									1					1
12														0
Totals	0	0	12	25	51	92	97	119	71	23	10	0	0	500

The means a and b are of course approximately* each equal to 6. Let y_M denote the value of the mean of the second throws corresponding to the value x of the first throw; then we have from the above table

x	1	2	3	4	5	6	7	8	9	10	11
y_M	3	4·6	4·9	5·2	5·7	6·1	6·6	7·0	7·5	8·0	8·0

These lie very nearly on the straight line

$$y_M - 6 = \tfrac{1}{2}(x - 6),$$

so

$$\frac{\sigma_2 r}{\sigma_1} = \tfrac{1}{2}.$$

Similarly if x_M denotes the value of the mean of the first throws corresponding to the value y of the second throw, we find very nearly

$$x_M - 6 = \tfrac{1}{2}(y - 6),$$

so

$$\frac{\sigma_1 r}{\sigma_2} = \tfrac{1}{2}.$$

Hence we have $\sigma_1 = \sigma_2$ and $r = \tfrac{1}{2}$, nearly. The value of σ_1 may be found from the frequency distribution of the first throws alone, ignoring second throws, which is (adding rows in the above table)

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* The computed values of the arithmetic means are $a = 5{\cdot}950$, $b = 6{\cdot}106$.

x 0	1	2	3	4	5	6	7	8	9	10	11	12
frequency 0	3	8	24	63	105	117	78	66	30	5	1	0

For simplicity, we shall assume that $a = b = 6$. We then obtain

x.	*Frequency.*	$(x-a)^2$.	*Product.*
0	0	36	
1	3	25	75
2	8	16	128
3	24	9	216
4	63	4	252
5	105	1	105
6	117	0	0
7	78	1	78
8	66	4	264
9	30	9	270
10	5	16	80
11	1	25	25
12	0	36	
	$n = 500$		$1493 = \Sigma(x-a)^2$

$$\sigma_1^2 = \frac{1}{n}\Sigma(x-a)^2 = 2{\cdot}986,$$

$$\sigma_1 = 1{\cdot}73.$$

The value of σ_2 is found in precisely the same way from the frequency distribution (adding columns in the above table)

y 0	1	2	3	4	5	6	7	8	9	10	11	12
frequency 0	0	12	25	51	92	97	119	71	23	10	0	0

We find $\sigma_2^2 = 2{\cdot}966$, so that $\sigma_2 = 1{\cdot}72$.

Lastly, let us compute the value of r from the formula of § 161, namely,

$$r = \frac{1}{n\sigma_1\sigma_2}\Sigma(x_1-a)(y_1-b).$$

We find $$\Sigma(x_1-a)(y_1-b) = 664,$$

and therefore $$\begin{aligned} r &= 0{\cdot}002 \times 0{\cdot}578 \times 0{\cdot}581 \times 664 \\ &= 0{\cdot}45, \end{aligned}$$

agreeing roughly with the previous determination.

Ex. 2.—When three of the 12 *dice were marked red, and were left down to be counted in the second throw, Darbishire's results were as follows:*

SECOND THROWS

FIRST THROWS	0	1	2	3	4	5	6	7	8	9	10	11	12
0													
1					1		1						
2							6	1					
3		1	1	5	2	2	4	5					
4			1	8	6	21	16	6	6				
5			4	3	12	15	23	22	9	3	1		
6		1		10	16	17	23	28	22	5	1		
7			1	4	9	17	18	24	16	5	3		
8				1	5	6	10	14	8	7	2	1	
9					4	3	9	6	6	2			
10						1	1	1	4	3			
11								1					
12													

Find the coefficient of correlation.

Ex. 3.—*When nine of the* 12 *dice were marked red, and were left down to be counted in the second throw, Darbishire's results were as follows:*

SECOND THROWS

FIRST THROWS	0	1	2	3	4	5	6	7	8	9	10	11	12
0													
1			1	1									
2			2	5	1	1							
3				5	7	3	1						
4			1	8	18	19	5	1					
5				6	17	30	32	13	1				
6				1	10	18	34	26	10	1			
7					4	17	26	30	18	7			
8							7	28	16	11	5		
9							3	6	15	9	7	1	
10								1		4	3	2	
11											1		1
12													

Find the coefficient of correlation.

Ex. 4.—*In the following table, which is due to Weldon,*[*] x *denotes the length of the carapace of the common shrimp,* y *denotes the length of the post-spinous portion of the carapace,* x_M *denotes the mean of the values of* x *corresponding to a definite value of* y, *and* y_M *denotes the mean of the values of* y *corresponding to a definite value of* x. *The mean value of* x *is* 249·63, *its standard deviation is* 6·73; *the mean value of* y *is* 177·53, *and its standard deviation is* 5·18.

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[*] *Proc. R.S.* **47** (1890), p. 445; **51** (1892), p. 2.

x.	y_x.
Over 260	188·41
260	185·41
259	183·25
258	182·25
257	182·34
256	182·22
255	181·14
254	179·98
253	179·50
252	179·17
251	178·68
250	177·71
249	177·39
248	176·64
247	175·36
246	175·20
245	173·56
244	173·31
243	173·33
242	172·81
241	171·30
240	169·57
Under 240	170·33

y.	x_y.
Over 186	262·11
186	258·25
185	256·15
184	256·84
183	254·88
182	254·18
181	253·28
180	251·73
179	251·34
178	249·78
177	249·10
176	248·53
175	246·79
174	245·73
173	245·02
172	243·89
171	243·67
170	241·28
169	241·06
Under 169	239·88

From these data show that $\frac{r\sigma_2}{\sigma_1} = 0{\cdot}67$, $\frac{\sigma_2}{r\sigma_1} = 0{\cdot}975$, *and consequently* $r = 0{\cdot}83$.

Ex. 5.*—*An urn containing white and black balls is so maintained that in drawing a ball the probability of getting a white ball is a constant* p, *and that of getting a black ball is* $q = 1 - p$. *The first drawing of a pair is to consist of* s *balls taken one at a time from the urn. The second drawing is to consist of* s *balls of which* t *are taken at random from the* s *first drawn, and* $s - t$ *are drawn one at a time from the urn. Show that the coefficient of correlation between the number of white balls in the first and second drawings of a pair is* t/s.

166. The Coefficient of Correlation for Frequency Distributions which are not Normal.

—The theory may be extended to frequency distributions which are not normal in the following way.†

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* H. L. Rietz, *Annals of Math.* **21** (1920), p. 306.

† Cf. G. U. Yule, *Proc. Roy. Soc.* **60** (1897), p. 477.

Let x and y denote deviations from the means of the measures of the two attributes. Let y_{M} denote as usual the mean of all the values of y which are observed in association with a given deviation x of the attribute A from its mean. Then in the case of normal frequency distributions we know (§ 162) that if the values of y_{M} are plotted against the corresponding values of x, the representative points lie on a straight line, namely, $y_{\text{M}} = bx$, where b has the value $\frac{\sigma_2 r}{\sigma_1}$.

In the case of non-normal frequency the points will not in general lie on a straight line, but let us try to find a constant b which will satisfy all the equations

$$y_{\text{M}1} = bx_1,$$
$$y_{\text{M}2} = bx_2,$$
$$\cdot \quad \cdot \quad \cdot$$

as well as possible, when $x_1, x_2, \ldots$ are the observed values of x, and $y_{\text{M}1}, y_{\text{M}2}, \ldots$ the associated values of y_{M}. We shall suppose that to the equation $y_{\text{M}r} = bx_r$ a weight is attached, equal to the number of observations on which it is based, say n_r. From these equations of condition we have at once the normal equation for b,

$$b\Sigma n_r x_r^2 = \Sigma n_r x_r y_{\text{M}r},$$

where the summation is over all the distinct values of x. This equation is evidently equivalent to the equation

$$b\Sigma x_r^2 = \Sigma x_r y_r,$$

where the summation is now not over all the distinct values of x, but over all the observations, so that the same value of x_r occurs n_r times in the sum.

If as before we write σ_1^2 for the mean of x^2, σ_2^2 for the mean of y^2, and $\sigma_1\sigma_2 r$ for the mean of xy, we see therefore that the straight line which best fits the points (x, y_{M}) is the straight line

$$y_{\text{M}} = \frac{\sigma_2 r}{\sigma_1} x,$$

just as in the case of normal frequency distributions. Similarly the straight line which best fits the points (x_M, y) is the straight line

$$x_M = \frac{\sigma_1 r}{\sigma_2} y.$$

We may then call the number r defined in this way the *coefficient of correlation* of the two attributes, even though the frequency distribution is not normal.

167. **The Correlation Ratio.**—A more satisfactory method of estimating the degree of correlation in a non-normal frequency distribution is by means of the *correlation ratio* * which is defined in the following way.

Let the total number of individual observations be N, and let $N\phi(x, y)dxdy$ be the number for which the attribute A lies between x and $x+dx$, while the attribute B lies between y and $y+dy$. Let $Nn_x dx$, where

$$n_x = \int_{-\infty}^{\infty} \phi(x, y)dy,$$

be the number of individuals for which A lies between x and $x+dx$, and let y_M denote the mean value of y for this set, so that

$$n_x y_M = \int_{-\infty}^{\infty} y\phi(x, y)dy.$$

There will be high correlation if the y's of this set are always clustered closely around the value y_M, *i.e.* if the standard deviation of these y's is always small. Denoting this standard deviation by σ_x, we have

$$n_x \sigma_x^2 = \int_{-\infty}^{\infty} (y - y_M)^2 \phi(x, y)dy.$$

If σ_x^2 is to be small for all values of x, its weighted average must be small. We shall denote this weighted average by θ^2, so

$$\theta^2 = \int_{-\infty}^{\infty} n_x \sigma_x^2 dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - y_M)^2 \phi(x, y)dxdy.$$

Now the standard deviation σ_2 is given by the formula

$$\sigma_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - b)^2 \phi(x, y)dxdy.$$

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* Cf. K. Pearson, "On the Theory of Skew Correlation," *Drapers' Research Mem.* **2** (1905).

If in this we write $\{(y-y_M)+(y_M-b)\}^2$ for $(y-b)^2$, we obtain

$$\sigma_2^2=\theta^2+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}2(y-y_M)(y_M-b)\phi(x,y)dxdy$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_M-b)^2\phi(x,y)dxdy.$$

Since $\int_{-\infty}^{\infty}(y-y_M)\phi(x,y)dy$ is zero, the first of these double integrals vanishes. So if we define a new number η by the equation

$$\eta^2=\frac{1}{\sigma_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_M-b)^2\phi(x,y)dxdy,$$

the last equation becomes

$$\eta^2=1-\frac{\theta^2}{\sigma_2^2}.$$

Since high correlation is associated with very small values of θ, we see that *high correlation is associated with values of η nearly equal to unity.* If, on the other hand, there is no correlation, there is no reason why the mean of the y's for each separate value of x should differ systematically from the mean of all the y's, and we may therefore expect $(y_M-b)^2$ to be small, and consequently η to be small.

The number η is called the *correlation ratio.** From the definition, we have

$$\eta^2=\frac{1}{\sigma_2^2}\int_{-\infty}^{\infty}n_x(y_M-b)^2dx,$$

so that η^2 is the weighted average of $(y_M-b)^2$ divided by σ_2^2.

168. Case of Normal Distributions.—We shall now show that when *the frequency distribution is normal the correlation ratio is identical with the correlation coefficient r.*

For in the case of normal frequency distributions, as we have seen (§ 162), we have

$$y_M-b=\frac{r\sigma_2}{\sigma_1}(x-a),$$

where a is the mean of all the x's,

so $$\eta^2=\frac{1}{\sigma_2^2}\left(\frac{r\sigma_2}{\sigma_1}\right)^2\int_{-\infty}^{\infty}n_x(x-a)^2dx.$$

---

* There is, of course, a second correlation ratio obtained by interchanging the parts played by x and y throughout.

Since $$\int_{-\infty}^{\infty} n_x(x-a)^2 dx = \sigma_1^2,$$
this gives $\eta^2 = r^2$, which establishes the proposition.

169. **Contingency Methods.**—It frequently happens that the attributes whose correlation we wish to discover are of such a nature that they do not admit of quantitative measurement—*e.g.* different colours,—and the groups into which they are classified cannot be arranged in a sequence possessing a logical order. To meet this case, what are known as *contingency methods* have been devised by K. Pearson.*

Let A represent any attribute, and let it be classified into groups $A_1, A_2, \ldots, A_s$; let the total number of individuals examined be N, and let the numbers which fall into these groups be $n_1, n_2, \ldots, n_s$ respectively. Then the probability of an individual falling into the pth group is $\frac{n_p}{N}$. Now let the same population be classified according to any other attribute into the groups $B_1, B_2, \ldots, B_t$, and let the group frequencies of the N individuals be $m_1, m_2, \ldots, m_t$ respectively, so that the probability of an individual falling into the qth group is $\frac{m_q}{N}$. Then by the theorem of Conjunctive Probability, if the two attributes were entirely uncorrelated, the probability of an individual falling into the group A_p and also into the group B_q would be $\frac{n_p m_q}{N^2}$, so the number of individuals to be expected satisfying these conditions would be $\frac{n_p m_q}{N}$, which we shall denote by ν_{pq}. Let the number actually observed as satisfying these conditions be n_{pq}. Then the differences $(n_{pq} - \nu_{pq})$, in so far as they are systematic, represent the correlation of the two attributes, and some function of them may be taken as a measure of the correlation. Pearson introduced two of these, namely, the *root-mean-square contingency* ϕ defined by the equation

$$\phi^2 = \frac{1}{N}\Sigma\frac{(n_{pq} - \nu_{pq})^2}{\nu_{pq}},$$

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* *Drapers' Co. Res. Mem.*, Biom. Series, i. (1904).

and the *mean contingency* ψ, defined by the equation

$$\psi = \frac{1}{\mathrm{N}}\Sigma'(n_{pq} - \nu_{pq}),$$

where Σ' denotes summation over the positive contingencies only.

170. **Case of Normal Distributions.**—We shall now show that *when the frequency distribution is normal, the root-mean-square contingency ϕ is connected with the correlation coefficient r by the equation*

$$\phi^2 = \frac{r^2}{1 - r^2}, \text{ or } \quad \text{arc sin } r = \text{arc tan } \phi.$$

For when the frequency distribution is normal, taking the origin of x and y at the centre of the distribution, we have (§ 162)

$$\frac{n_p}{\mathrm{N}} = \frac{1}{\sigma_1\sqrt{(2\pi)}}e^{-\frac{x^2}{2\sigma_1^2}}, \qquad \frac{m_q}{\mathrm{N}} = \frac{1}{\sigma_2\sqrt{(2\pi)}}e^{-\frac{y^2}{2\sigma_2^2}}$$

so
$$\nu_{pq} = \frac{\mathrm{N}}{2\pi\sigma_1\sigma_2}e^{-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}}$$

and
$$n_{pq} = \frac{\mathrm{N}}{2\pi\sigma_1\sigma_2\sqrt{(1 - r^2)}}e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)}.$$

Therefore

$$\frac{(n_{pq} - \nu_{pq})^2}{\mathrm{N}\nu_{pq}} = \frac{1}{2\pi\sigma_1\sigma_2}\left\{\begin{array}{l}\frac{1}{1 - r^2}e^{-\frac{1+r^2}{2(1-r^2)}\frac{x^2}{\sigma_1^2}+\frac{2rxy}{\sigma_1\sigma_2(1-r^2)}-\frac{1+r^2}{2(1-r^2)}\cdot\frac{y^2}{\sigma_2^2}} \\ -\frac{2}{\sqrt{(1 - r^2)}}e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)} + e^{-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}}\end{array}\right\}.$$

Substituting this expression in $\phi^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(n_{pq} - \nu_{pq})^2}{\mathrm{N}\nu_{pq}}dxdy$, and remembering that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(ax^2+2hxy+by^2)}dxdy = \frac{\pi}{\sqrt{(ab - h^2)}},$$

we have
$$\phi^2 = \frac{1}{1 - r^2} - 2 + 1 = \frac{r^2}{1 - r^2},$$

which is the required result.[*]

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[*] For other deductions, cf. W. P. Elderton, *Frequency Curves and Correlation* (London, 1906), p. 148.

171. **Multiple Normal Correlation.**—We shall now extend the theory to the case when more attributes than two are considered. For simplicity we shall suppose the number to be three, but the formulae admit of an obvious generalisation to the case of any number. The frequency distribution will be supposed to be normal, so the probability that the attribute α has a measure between x and $x+dx$, while the attribute β has a measure between y and $y+dy$, and the attribute γ has a measure between z and $z+dz$, is $\phi(x, y, z)dxdydz$, where

$$\phi(x, y, z) = \mathrm{K}e^{-(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)}, \tag{1}$$

and K, a, b, c, f, g, h denote constants, the origin having been taken so that the mean values of x, y, and z are zero. The constant K may be determined at once from the equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x, y, z)dxdydz = 1\,;$$

for, remembering that if $\mathrm{F}(x_1, x_2, \ldots, x_n)$ is a positive quadratic form and Δ its determinant, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-\mathrm{F}(x_1, x_2, \ldots, x_n)}dx_1dx_2 \ldots dx_n = \frac{\pi^{\frac{1}{2}n}}{\Delta^{\frac{1}{2}}},$$

we have

$$\mathrm{K} = \frac{\sqrt{\Delta}}{\pi^{\frac{3}{2}}}, \tag{2}$$

where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2. \tag{3}$$

By integrating the expression (1) we readily obtain the following results:

The probability that α is between x and $x+dx$, while β is between y and $y+dy$, γ being disregarded, is

$$\frac{1}{\pi}\frac{\Delta^{\frac{1}{2}}}{c^{\frac{1}{2}}}e^{-\frac{1}{c}(\mathrm{B}x^2+\mathrm{A}y^2-2\mathrm{H}xy)}dxdy, \tag{4}$$

where A, B, . . . are the co-factors of a, b, . . . in the determinant Δ.

Comparing (4) with the usual form of the frequency function for two variables, namely,

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}}e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2rxy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)}dxdy,$$

we have $$r=\frac{\mathrm{H}}{\sqrt{(\mathrm{AB})}},\quad \sigma_1^2=\frac{\mathrm{A}}{2\Delta},\quad \sigma_2^2=\frac{\mathrm{B}}{2\Delta}.$$

The coefficient of correlation of the attributes α *and* β *is therefore* $\frac{\mathrm{H}}{\sqrt{(\mathrm{AB})}}$. We shall denote this by r_{12}. Introducing similarly the coefficient of correlation r_{23} between β and γ, and r_{13} between α and γ, let us consider the determinant

$$\mathrm{R}=\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}$$

and denote by R_{pq} the co-factor of the element in the pth row and qth column. We have

$$\mathrm{R}=\begin{vmatrix} 1 & \frac{\mathrm{H}}{\sqrt{(\mathrm{AB})}} & \frac{\mathrm{G}}{\sqrt{(\mathrm{AC})}} \\ \frac{\mathrm{H}}{\sqrt{(\mathrm{AB})}} & 1 & \frac{\mathrm{F}}{\sqrt{(\mathrm{BC})}} \\ \frac{\mathrm{G}}{\sqrt{(\mathrm{AC})}} & \frac{\mathrm{F}}{\sqrt{(\mathrm{BC})}} & 1 \end{vmatrix}=\frac{\Delta^2}{\mathrm{ABC}},$$

$$\mathrm{R}_{11}=\frac{a\Delta}{\mathrm{BC}},\quad \mathrm{R}_{12}=\frac{h\Delta}{\mathrm{C}\sqrt{(\mathrm{AB})}},\quad \mathrm{R}_{13}=\frac{g\Delta}{\mathrm{B}\sqrt{(\mathrm{AC})}},\text{ etc.}$$

These equations may be written

$$\mathrm{R}=\frac{1}{8\Delta\sigma_1^2\sigma_2^2\sigma_3^2},\quad \mathrm{R}_{11}=2\mathrm{R}\sigma_1^2 a,\quad \mathrm{R}_{12}=2\mathrm{R}\sigma_1\sigma_2 h,\text{ etc.,}$$

so we have $$a=\frac{\mathrm{R}_{11}}{2\mathrm{R}\sigma_1^2},\quad h=\frac{\mathrm{R}_{12}}{2\mathrm{R}\sigma_1\sigma_2},\text{ etc.,}$$

and thus finally expressing the frequency function in terms of the correlation coefficients and standard deviations, *the probability that the attribute* α *has a measure between* x *and* $x+dx$, *while* β *has a measure between* y *and* $y+dy$, *and* γ *has a measure between* z *and* $z+dz$, *is*

$$\phi(x,y,z)\,dxdydz,$$

where

$$\phi(x, y, z) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1\sigma_2\sigma_3 R^{\frac{1}{2}}} e^{-\frac{1}{2R}\left(\frac{R_{11}x^2}{\sigma_1^2}+\frac{R_{22}y^2}{\sigma_2^2}+\frac{R_{33}z^2}{\sigma_3^2}+\frac{2R_{23}yz}{\sigma_2\sigma_3}+\frac{2R_{31}zx}{\sigma_3\sigma_1}+\frac{2R_{12}xy}{\sigma_1\sigma_2}\right)}.$$

Similarly in the general case* *when there are n variables* $x_1, x_2, \ldots, x_n$ *whose standard deviations are* $\sigma_1, \sigma_2, \ldots, \sigma_n$, *and whose correlation coefficients in pairs are* $r_{12}, r_{23}, \ldots$, *the probability that the first attribute has a measure between* x_1 *and* $x_1 + dx_1$, *while the second attribute has a measure between* x_2 *and* $x_2 + dx_2$, *and so on, is*

$$\phi(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n,$$

where

$$\phi = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma_1\sigma_2 \ldots \sigma_n R^{\frac{1}{2}}} e^{-\frac{1}{2R}\left(\frac{R_{11}x_1^2}{\sigma_1^2}+\frac{R_{22}x_2^2}{\sigma_2^2}+\ldots+\frac{2R_{12}x_1x_2}{\sigma_1\sigma_2}+\ldots\right)}.$$

Here R denotes the determinant $\begin{vmatrix} 1 & r_{12} & r_{13} & \ldots & r_{1n} \\ r_{21} & 1 & r_{23} & \ldots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & r_{n3} & \ldots & 1 \end{vmatrix}$, and R_{pq} denotes the co-factor of the element in the pth row and qth column.

Ex. 1.—*In the case of three attributes, suppose that* γ *is known to have a measure between* z *and* $z + dz$: *show that the probability that* α *has a measure between* x *and* $x + dx$, *while* β *has a measure between* y *and* $y + dy$, *is*

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{R}} e^{-\frac{R_{11}}{2R\sigma_1^2}\left(x-\frac{r_{13}\sigma_1}{\sigma_3}z\right)^2-\frac{R_{22}}{2R\sigma_2^2}\left(y-\frac{r_{23}\sigma_2}{\sigma_3}z\right)^2-\frac{R_{12}}{R\sigma_1\sigma_2}\left(x-\frac{r_{13}\sigma_1}{\sigma_3}z\right)\left(y-\frac{r_{23}\sigma_2}{\sigma_3}z\right)} dx dy,$$

so that *when the measure of* γ *is known to be* z, *the mean value of* α *is* $\frac{r_{13}\sigma_1}{\sigma_3}z$, *and the mean value of* β *is* $\frac{r_{23}\sigma_2}{\sigma_3}z$.

Ex. 2.—*In the case of three attributes, suppose that* γ *is known to have a measure between* z *and* $z + dz$, *and* β *to have a measure between* y *and* $y + dy$. *Show that the probability that* α *has a measure between* x *and* $x + dx$ *is*

$$\frac{R_{11}^{\frac{1}{2}}}{(2\pi R)^{\frac{1}{2}}\sigma_1} e^{-\frac{R_{11}}{2R\sigma_1^2}\left(x+\frac{\sigma_1 R_{12}}{\sigma_2 R_{11}}y+\frac{\sigma_1 R_{13}}{\sigma_3 R_{11}}z\right)^2} dx,$$

so that *when the measure of* γ *is known to be* z, *and the measure of* β *is known to be* y, *the mean value of the measure of* α *is*

$$-\frac{\sigma_1 R_{12}}{\sigma_2 R_{11}}y - \frac{\sigma_1 R_{13}}{\sigma_3 R_{11}}z.$$

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* K. Pearson, *Phil. Trans.* **187 A** (1896), p. 253; **200 A** (1902), p. 1.

CHAPTER XIII

THE SEARCH FOR PERIODICITIES

172. Introduction.—In Chapter X. we have been concerned with sums of trigonometric terms of the type

$$a_1 \cos(nt + \epsilon_1) + a_2 \cos(2nt + \epsilon_2) + a_3 \cos(3nt + \epsilon_3) + \ldots \quad (1)$$

As explained in § 132, the vibration of a violin string may be represented by a series of this kind when t denotes the time and $a_1, a_2, \ldots$ are certain functions of position on the string, the individual terms of the series corresponding to the fundamental note of the string and its various overtones.

It is shown in works on the Theory of Sound that if instead of a violin string we consider a bar vibrating laterally (*e.g.* a tuning fork), we obtain for the motion at a definite point of the bar a series of the type

$$a_1 \cos(n_1 t + \epsilon_1) + a_2 \cos(n_2 t + \epsilon_2) + a_3 \cos(n_3 t + \epsilon_3) + \ldots \quad (2)$$

where $a_1, a_2, a_3, \ldots$ are certain functions of position on the bar, but where we now no longer have n_2 equal to twice n_1, or n_3 equal to three times n_1; in fact, the ratios $n_1 : n_2 : n_3 : \ldots$ are equal to the ratios of the squares of the roots of the equation $\cos m \cosh m + 1 = 0$, so that $n_1 : n_2 : n_3 : \ldots = 3{\cdot}52 \ldots : 22{\cdot}03 \ldots : 61{\cdot}70 \ldots : \ldots$ The sum of a series of the type (2) is evidently not a periodic function of t, but we can speak of it as constituted of elements which are periodic, the periods being $\frac{2\pi}{n_1}, \frac{2\pi}{n_2}$, etc.

In many branches of physical science, especially in meteorology and astronomy, phenomena are observed which may be represented by sums resembling (2): for example, the height of sea-water at any instant depends on a number of *constituent* *tides*, of which one (the semi-diurnal tide, which is the largest constituent tide along the British coasts) has a period of half a day, another (the diurnal tide) has a period of a day, another (the fortnightly tide) has a period of nearly a fortnight, and so on. Each constituent tide produces its own effect independently of the others, and the actual height of water is the sum of these effects. The height of water can therefore be represented by an expression of the form

$$y = a_0 + a_1 \cos(n_1 t + \epsilon_1) + a_2 \cos(n_2 t + \epsilon_2) + \ldots + a_k \cos(n_k t + \epsilon_k),$$

each of the trigonometrical terms corresponding to one of the constituent tides.

In the case of other phenomena, *e.g.* the spottedness of the sun, the variation of the observed quantity appears to consist of an accidental or capricious part, which cannot be represented by any analytical expression, superposed on a systematic part, which the mathematician may attempt to represent by an expression of the form (2). The area covered by spots on the sun certainly fluctuates in a way which suggests a certain amount of regularity in the variation, maxima occurring at intervals of (on the average) rather more than eleven years.

If a series of observations of any quantity are taken, and there is reason to expect that they can be represented by a sum of trigonometric terms, or that they involve (entangled with an irregular variation) a regular variation which can be represented by a sum of trigonometric terms, then the first task of the mathematician is *to discover the periods* $\frac{2\pi}{n_1}, \frac{2\pi}{n_2}, \ldots$ of these constituent terms. This must always be done before we attempt to find the amplitudes $a_1, a_2, a_3, \ldots$ or the phases $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ In many cases the periods are known *a priori* from theory or from some reasonable ground of expectation: for instance, we should naturally expect the periods of the regular terms in the temperature at a given place to be a day and a year. But in many other cases, *e.g.* the spottedness of the sun, the periods are quite unknown. We shall show in the present chapter how they may be discovered.

173. Testing for an Assumed Period.—Let the observed measures of the phenomenon, made at equal intervals of time, be denoted by

$$u_0, u_1, u_2, u_3, \ldots \tag{1}$$

and suppose that it is desired to test this sequence for a periodicity whose period extends over p consecutive numbers of the sequence; u_x might, for instance, mean the number of earthquakes in the year x, and we might wish to know whether the liability to earthquakes is greater every p years. Let v_r denote the remainder when an integer r is divided bv p, so that the sequence

$$v_0, v_1, v_2, v_3, \ldots \tag{2}$$

is simply

$$0, 1, 2, \ldots (p-1), 0, 1, 2, \ldots (p-1), 0, 1, 2, \ldots \tag{3}$$

Then the question "Does the sequence (1) involve the assumed periodicity?" may be expressed more precisely thus: "*Does correlation exist between the sequence* (1) *and the sequence* (2)*?*" As the frequency distribution with which we are dealing is not likely to be normal, the correlation ratio (§ 167) is a better method of estimating correlation than the correlation coefficient. To find the correlation ratio, we must first arrange the u's in columns, so that all the u's which correspond to the same value of v are in the same column. This may obviously be done by merely writing down the u's in order in horizontal lines, each of which contains p u's thus:

$$\begin{array}{llllll}
 & u_0 & u_1 & u_2 & \cdots & u_{p-1} \\
 & u_p & u_{p+1} & u_{p+2} & \cdots & u_{2p-1} \\
 & u_{2p} & u_{2p+1} & u_{2p+2} & \cdots & u_{3p-1} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & u_{(m-1)p} & u_{(m-1)p+1} & u_{(m-1)p+2} & \cdots & u_{mp-1} \\
\cline{2-6}
\text{Sums} & U_0 & U_1 & U_2 & & U_{p-1}
\end{array}$$

All the u's in the first column correspond to the value zero of v, all the u's in the second column correspond to the value unity of v, and so on: we have taken enough of the observational material to fill m horizontal rows, and we have denoted the sums of the individual columns by $U_0, U_1, \ldots, U_{p-1}$.

Dividing these last numbers by m we obtain the means M_0, M_1, . . ., M_{p-1} of the values of u in the individual columns. Then (§ 167) *the correlation ratio η is the standard deviation of the M's, divided by the standard deviation of the u's.* The value of η is calculated in this way for a large number of values of p, and the results plotted as a curve in which p is the abscissa and the corresponding value of η is the ordinate. This curve will be called a *periodogram.**

It is easy to see why the ratio of the standard deviation of the M's to the standard deviation of the u's is a suitable indicator of periodicity. For in the course of one horizontal row of the above scheme, the part of the phenomenon which is of period p will pass through all the phases of one complete period, so that this periodic part is in the same phase at all terms which are above or below each other in the same vertical column, *e.g.* it is in the same phase at the terms u_r, u_{p+r}, u_{2p+r}, . . ., $u_{(m-1)p+r}$. The part of the phenomenon which is of period p therefore appears with m-fold amplitude in the row U_0, U_1, . . ., U_{p-1}, and therefore appears with its own proper amplitude in the row of means M_0, M_1, . . ., M_{p-1}. Any accidental disturbance on the other hand, or any periodic disturbance of period different from p, will be enfeebled by the process of forming means, since positive and negative deviations will tend to annul each other; and therefore, when a periodicity of period p exists, the standard deviation of the M's has a value much larger than when a periodicity of this period does not exist in the phenomenon.

174. **The Periodogram in the Neighbourhood of a True Period.**—Suppose now that each of the terms of the sequence u_x consists of a simple periodic part of period T, say $a \sin \frac{2\pi x}{T}$, together with a part which does not involve this periodicity, say b_x, so

$$u_x = a \sin \frac{2\pi x}{T} + b_x.$$

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* The term *periodogram* was introduced by Schuster, *Terrestrial Magnetism*, **3** (1898), p. 24. Schuster's periodogram differs from that introduced above, but the similarity of form and purpose is so great that it has seemed best to retain the name.

Denote by σ_b the standard deviation of the b's, and denote by σ the standard deviation of the u's. Since the standard deviation of the sequence $0, \sin\frac{2\pi}{T}, \sin\frac{4\pi}{T}, \sin\frac{6\pi}{T}, \ldots$, is $\frac{1}{\sqrt{2}}$, and there is no correlation between b_x and $a\sin\frac{2\pi x}{T}$, we have

$$\sigma^2 = \tfrac{1}{2}a^2 + \sigma_b^2.$$

Next, let U_x denote as before the sum of $u_x + u_{p+x} + \ldots + u_{(m-1)p+x}$, and let B_x denote the sum $b_x + b_{p+x} + \ldots + b_{(m-1)p+x}$. Then

$$U_x = a\left\{\sin\frac{2\pi x}{T} + \sin\frac{2\pi(p+x)}{T} + \ldots + \frac{\sin 2\pi(m-1)p + 2\pi x}{T}\right\} + B_x$$

or

$$U_x = a\frac{\sin\frac{m\pi p}{T}}{\sin\frac{\pi p}{T}}\sin\left\{\frac{2\pi x}{T} + \frac{(m-1)\pi p}{T}\right\} + B_x.$$

Denote by Σ the standard deviation of the U's and by Σ_B the standard deviation of the B's. Then in the same way as we found σ, we find

$$\Sigma^2 = \frac{1}{2}a^2\frac{\sin^2\frac{m\pi p}{T}}{\sin^2\frac{\pi p}{T}} + \Sigma_B^2.$$

Now, since B_x is the sum of m of the b_x's, we may write $\Sigma_B^2 = m\sigma_b^2$, and therefore

$$\Sigma^2 = \frac{1}{2}a^2\frac{\sin^2\frac{m\pi p}{T}}{\sin^2\frac{\pi p}{T}} + m\sigma_b^2.$$

Thus if Σ_M denotes the standard deviation of the means $M_0, M_1, \ldots, M_m$ of the individual columns, we have

$$\Sigma_M^2 = \frac{1}{m^2}\Sigma^2$$

$$= \frac{a^2}{2m^2}\frac{\sin^2\frac{m\pi p}{T}}{\sin^2\frac{\pi p}{T}} + \frac{1}{m}\sigma_b^2.$$

Therefore, if η denotes as usual the correlation ratio, we have

$$\eta^2 = \frac{\Sigma_M^2}{\sigma^2}$$

$$= \frac{\frac{a^2}{2m^2}\frac{\sin^2\frac{m\pi p}{T}}{\sin^2\frac{\pi p}{T}} + \frac{1}{m}\sigma_b^2}{\frac{1}{2}a^2 + \sigma_b^2}.$$

This is the equation of the periodogram, when p and η are taken as rectangular co-ordinates; or to speak more accurately, it is the form to which the equation of the periodogram tends as the amount of observational material, used in constructing it, is increased indefinitely. The number m will, in most cases, be taken greater than 20 if there is sufficient observational material to provide so many horizontal rows.

From the above equation it is obvious that with such a value of m, η is a rather small fraction, except when p is nearly equal to T. Let $p = T(1 - \epsilon)$, where ϵ is a small number; then as ϵ tends to zero, the above value of η^2 tends to the value

$$\frac{\frac{1}{2}a^2 + \frac{1}{m}\sigma_b^2}{\frac{1}{2}a^2 + \sigma_b^2},$$

and as m is a large number, this is nearly $\dfrac{1}{1 + \dfrac{2\sigma_b^2}{a^2}}$. It falls away rapidly as ϵ passes away from zero in either direction, and when $\epsilon = \pm\frac{1}{m}$, η^2 becomes

$$\frac{\frac{1}{m}\sigma_b^2}{\frac{1}{2}a^2 + \sigma_b^2},$$

which is the smallest value it can take for any value of p. There are maxima of η again near the values of p given by $\frac{m\pi p}{T} = m\pi \pm \frac{3\pi}{2}$, that is when ϵ is nearly equal to $\pm\frac{3}{2m}$. Collecting our results, we may say that *when the phenomenon studied is a simple periodic disturbance of period* T, *superposed on a non-systematic disturbance, and the periodogram is computed with a large value of* m, *the periodogram curve is close to the axis of* p *except when* p *is in the neighbourhood of* T, *where the curve has a peak of breadth* $\frac{2T}{m}$, *flanked by smaller peaks on both sides.* The recognition of these peaks in the periodogram is the means by which we discover hidden periodicities.

175. **An Example of Periodogram Analysis.**—The table below gives the magnitude (*i.e.* a measure of the brightness) of a variable star at midnight on 600 successive days. (These magnitudes were obtained by reading off from a curve, on which all the observations of the star's brightness were plotted: they have been reduced to a scale suitable for periodogram analysis.) It is required to find a trigonometrical function which will represent the magnitude at any time t.

Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.
1	25	21	2	41	10	61	27	81	17
2	28	22	4	42	7	62	25	82	18
3	31	23	8	43	5	63	24	83	19
4	32	24	11	44	3	64	21	84	19
5	33	25	15	45	3	65	19	85	19
6	33	26	19	46	3	66	17	86	19
7	32	27	23	47	4	67	15	87	20
8	31	28	26	48	5	68	13	88	20
9	28	29	29	49	7	69	12	89	20
10	25	30	32	50	10	70	11	90	20
11	22	31	33	51	13	71	11	91	20
12	18	32	34	52	16	72	10	92	20
13	14	33	33	53	19	73	10	93	20
14	10	34	32	54	22	74	11	94	20
15	7	35	30	55	24	75	12	95	21
16	4	36	27	56	26	76	12	96	20
17	2	37	24	57	27	77	13	97	20
18	0	38	20	58	28	78	14	98	20
19	0	39	17	59	29	79	15	99	20
20	0	40	13	60	28	80	16	100	19

Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.
101	18	142	12	183	19	224	15	265	27
102	17	143	16	184	15	225	15	266	29
103	16	144	19	185	12	226	16	267	30
104	15	145	23	186	9	227	17	268	30
105	13	146	27	187	7	228	17	269	30
106	12	147	30	188	5	229	17	270	29
107	11	148	32	189	4	230	17	271	27
108	10	149	33	190	4	231	18	272	25
109	9	150	34	191	5	232	18	273	22
110	9	151	33	192	5	233	19	274	19
111	10	152	32	193	7	234	19	275	16
112	10	153	30	194	9	235	20	276	12
113	11	154	27	195	12	236	20	277	9
114	12	155	24	196	14	237	21	278	6
115	14	156	20	197	17	238	21	279	4
116	16	157	16	198	20	239	22	280	2
117	19	158	12	199	22	240	22	281	1
118	21	159	9	200	24	241	22	282	1
119	24	160	5	201	25	242	22	283	2
120	25	161	3	202	26	243	22	284	4
121	27	162	1	203	27	244	21	285	7
122	28	163	0	204	27	245	20	286	10
123	29	164	0	205	26	246	19	287	14
124	29	165	1	206	25	247	17	288	17
125	28	166	3	207	24	248	16	289	21
126	27	167	6	208	22	249	14	290	25
127	25	168	9	209	20	250	12	291	29
128	23	169	13	210	18	251	11	292	31
129	20	170	17	211	17	252	9	293	33
130	17	171	21	212	15	253	8	294	34
131	14	172	24	213	14	254	7	295	34
132	11	173	27	214	13	255	8	296	33
133	8	174	30	215	13	256	8	297	31
134	5	175	32	216	12	257	9	298	29
135	4	176	33	217	12	258	10	299	26
136	2	177	33	218	12	259	12	300	22
137	2	178	32	219	13	260	14	301	19
138	2	179	31	220	13	261	17	302	14
139	4	180	28	221	13	262	20	303	11
140	6	181	25	222	14	263	23	304	7
141	9	182	22	223	14	264	25	305	4

Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.
306	2	347	25	388	23	429	5	470	24
307	1	348	25	389	22	430	8	471	22
308	0	349	25	390	21	431	12	472	19
309	1	350	24	391	19	432	15	473	16
310	2	351	24	392	17	433	19	474	13
311	5	352	22	393	15	434	23	475	11
312	7	353	21	394	13	435	27	476	9
313	11	354	19	395	11	436	30	477	8
314	15	355	18	396	9	437	32	478	7
315	19	356	17	397	7	438	34	479	7
316	22	357	16	398	6	439	34	480	7
317	25	358	15	399	6	440	34	481	8
318	28	359	15	400	6	441	32	482	9
319	30	360	14	401	7	442	30	483	11
320	32	361	14	402	8	443	28	484	12
321	32	362	14	403	10	444	24	485	14
322	32	363	14	404	12	445	20	486	16
323	31	364	14	405	15	446	16	487	18
324	29	365	14	406	18	447	13	488	20
325	26	366	14	407	22	448	9	489	21
326	23	367	14	408	24	449	6	490	22
327	21	368	14	409	27	450	3	491	23
328	17	369	14	410	29	451	2	492	23
329	14	370	14	411	31	452	1	493	23
330	11	371	15	412	31	453	1	494	23
331	9	372	15	413	31	454	2	495	23
332	7	373	15	414	31	455	4	496	22
333	6	374	15	415	29	456	6	497	21
334	5	375	16	416	27	457	9	498	20
335	6	376	16	417	24	458	13	499	19
336	6	377	17	418	21	459	17	500	18
337	7	378	18	419	18	460	20	501	18
338	9	379	19	420	14	461	23	502	17
339	11	380	20	421	10	462	26	503	17
340	13	381	21	422	7	463	28	504	16
341	15	382	22	423	5	464	30	505	16
342	18	383	23	424	2	465	31	506	16
343	20	384	23	425	1	466	31	507	16
344	22	385	24	426	0	467	31	508	15
345	23	386	24	427	1	468	29	509	15
346	24	387	24	428	2	469	27	510	15

Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.	Day.	Mag.
511	14	529	25	547	8	565	12	583	34
512	14	530	26	548	10	566	8	584	34
513	13	531	26	549	13	567	6	585	33
514	13	532	25	550	16	568	3	586	31
515	13	533	24	551	20	569	1	587	29
516	13	534	23	552	23	570	0	588	26
517	13	535	21	553	26	571	0	589	22
518	13	536	19	554	29	572	1	590	18
519	14	537	16	555	31	573	3	591	15
520	14	538	14	556	32	574	6	592	11
521	15	539	12	557	32	575	10	593	8
522	16	540	9	558	32	576	13	594	5
523	18	541	7	559	31	577	17	595	3
524	19	542	5	560	29	578	21	596	2
525	21	543	5	561	26	579	25	597	2
526	22	544	4	562	23	580	28	598	2
527	24	545	5	563	20	581	31	599	4
528	24	546	6	564	16	582	33	600	5

We have first to find the mean value and standard deviation of the observations. By the methods of Chapter VIII. we find

$$\text{Mean value} = 17,$$
$$\text{Standard deviation} = 8{\cdot}63.$$

As there are on the whole about 21 maxima and 21 minima in the 600 days, we suspect that one of the most important periods will be not far from 600/21 days. We shall therefore take *trial periods* ranging from 20 days to $32\frac{1}{2}$ days; that is, we shall give to p in succession the values 20, $20\frac{1}{2}$, 21, $21\frac{1}{2}$, and so on to $32\frac{1}{2}$. Taking $m = 17$, the summation process for, *e.g.*, 24 days is as follows:

25	28	31	32	33	33	32	31	28	25	22	18	14	10	7	4	2	0	0	0	2	4	8	11
15	19	23	26	29	32	33	34	33	32	30	27	24	20	17	13	10	7	5	3	3	3	4	5
7	10	13	16	19	22	24	26	27	28	29	28	27	25	24	21	19	17	15	13	12	11	11	10
10	11	12	12	13	14	15	16	17	18	19	19	19	19	20	20	20	20	20	20	20	20	21	20
20	20	20	19	18	17	16	15	13	12	11	10	9	9	10	10	11	12	14	16	19	21	24	25
27	28	29	29	28	27	25	23	20	17	14	11	8	5	4	2	2	2	4	6	9	12	16	19
23	27	30	32	33	34	33	32	30	27	24	20	16	12	9	5	3	1	0	0	1	3	6	9
13	17	21	24	27	30	32	33	33	32	31	28	25	22	19	15	12	9	7	5	4	4	5	5
7	9	12	14	17	20	22	24	25	26	27	27	26	25	24	22	20	18	17	15	14	13	13	12
12	12	13	13	13	14	14	15	15	16	17	17	17	17	18	18	19	19	20	20	21	21	22	22
22	22	22	21	20	19	17	16	14	12	11	9	8	7	8	8	9	10	12	14	17	20	23	25
27	29	30	30	30	29	27	25	22	19	16	12	9	6	4	2	1	1	2	4	7	10	14	17
21	25	29	31	33	34	34	33	31	29	26	22	19	14	11	7	4	2	1	0	1	2	5	7
11	15	19	22	25	28	30	32	32	32	31	29	26	23	21	17	14	11	9	7	6	5	6	6
7	9	11	13	15	18	20	22	23	24	25	25	25	24	24	22	21	19	18	17	16	15	15	14
14	14	14	14	14	14	14	14	14	14	15	15	15	15	16	16	17	18	19	20	21	22	23	23
24	24	24	23	22	21	19	17	15	13	11	9	7	6	6	6	7	8	10	12	15	18	22	24
285	319	353	371	389	406	407	408	392	376	359	326	294	259	242	208	191	174	173	172	188	204	238	254

The sums at the foot of the columns are the numbers U_0, U_1, U_2, . . . corresponding to this trial period.

When the trial period is not a whole number of days, we modify the arrangement slightly so as to secure that terms in the same phase are still in the same vertical column: thus if the trial period were 31½ days, we should write the values corresponding to days 1 to 31 in the first horizontal row, and the values corresponding to days 32 to 62 in the second horizontal row, then we should omit altogether the value corresponding to day 63 in order to bring the value corresponding to day 64 to the beginning of the third row, and so on.

The table of values of the u's obtained by the summation process with the different trial periods is as follows:

Period (p) (days).

20.	20½.	21.	21½.	22.	22½.	23.	23½.	24.	24½.	25.	25½.	26.	26½.	27.	27½.	28.	28½.	29.	29½.	30.	30½.	31.	31½.	32.	32½.
292	274	285	282	285	283	251	211	285	359	333	295	294	290	301	281	227	282	437	442	337	291	315	324	300	293
291	278	290	285	285	287	252	227	319	373	330	297	297	295	310	283	240	318	454	431	320	290	315	323	296	295
297	281	295	286	293	288	254	247	353	381	326	298	304	300	315	284	261	350	472	419	307	284	315	318	287	296
292	284	297	288	295	291	258	272	371	384	317	299	304	305	318	285	272	379	471	396	284	282	314	312	287	295
291	288	301	291	305	296	264	297	389	379	308	301	307	310	321	287	289	405	471	373	270	277	311	307	282	296
293	293	305	292	309	296	273	321	406	371	299	300	306	312	322	287	306	424	452	344	249	274	308	299	277	295
297	297	305	295	308	299	284	344	407	354	291	300	307	314	319	288	328	437	432	313	234	273	304	291	279	296
293	299	307	296	311	302	296	362	408	337	281	301	303	315	319	290	338	445	412	280	222	271	300	284	270	294
292	303	308	299	315	302	307	379	392	313	272	297	303	314	315	290	351	443	375	248	216	271	294	275	267	296
293	306	304	301	314	301	317	386	376	291	266	294	296	316	308	289	361	434	339	220	206	270	289	268	266	294
298	306	306	305	315	302	326	390	359	267	263	294	297	312	305	292	374	420	303	193	208	273	285	253	271	292
292	306	304	306	312	301	333	385	326	245	260	291	290	310	298	292	371	395	265	172	205	272	279	257	266	290
292	305	293	305	311	298	338	375	294	228	259	287	290	307	289	293	371	371	229	153	210	278	275	253	270	289
290	304	297	303	305	298	340	358	259	214	262	286	285	303	284	294	367	342	194	144	219	279	272	252	271	289
296	300	293	300	303	296	338	337	242	209	267	284	288	300	278	296	365	310	161	139	236	286	269	252	281	287
287	297	287	298	292	292	332	314	208	205	273	281	282	295	270	298	349	278	143	146	247	289	267	253	280	287
287	295	287	295	287	291	324	287	191	212	279	280	286	291	267	298	336	246	127	155	268	295	266	258	285	286
284	287	282	291	276	288	313	261	174	220	286	278	283	286	263	301	320	215	112	173	282	299	267	263	291	285
289	285	279	288	274	285	302	237	173	237	295	279	288	283	259	303	308	189	115	195	301	304	268	269	302	285
282	281	279	283	266	285	289	215	172	254	302	280	287	279	260	302	284	168	118	222	319	309	271	278	301	284
		277	279	267	284	276	197	188	276	307	282	292	275	261	302	267	151	137	253	341	312	275	286	307	285
				262	282	263	189	204	298	310	283	291	272	261	303	249	141	157	285	352	316	280	295	311	286
						252	186	238	318	314	287	296	270	268	302	240	140	194	320	369	315	284	306	319	287
								254	337	314	287	295	268	272	299	222	144	231	349	373	318	289	312	316	288
										312	290	297	267	277	299	212	156	267	379	378	313	294	320	216	289
												296	266	285	297	206	175	301	400	379	313	299	327	315	290
														293	291	209	198	336	421	380	309	303	330	319	291
																205	225	369	431	363	304	308	332	309	292
																		403	442	361	297	310	334	304	294
																				343	292	313	331	299	294
																						314	327	298	296
																								286	296

We have next to find the standard deviation of each of these columns and divide it by m ($=17$) in order to obtain the standard deviation of the corresponding column of means $M_0, M_1, M_2, \ldots$.

Now dividing these standard deviations of the M's by the standard deviation of the u's, which was found to be 8·63, we have a table of values of the correlation-ratio η corresponding to the different values of the trial period p. The values of η may evidently be obtained at once by dividing the corresponding standard deviations of the columns by 146·71 ($=17 \times 8{\cdot}63$). The results are as follow:

Period.	Arithmetic Mean.	Standard Deviation.	η.	Period.	Arithmetic Mean.	Standard Deviation.	η.
20	291·4	4·042	0·028	26½	294·4	16·837	0·115
20½	293·5	10·171	0·069	27	290·3	22·132	0·151
21	294·6	10·018	0·068	27½	293·6	6·590	0·045
21½	293·7	7·947	0·054	28	293·9	58·197	0·397
22	295	17·034	0·116	28½	292·2	109·019	0·743
22½	293	6·772	0·046	29	292·3	126·481	0·862
23	294·9	31·630	0·216	29½	291	106·089	0·723
23½	294·7	69·389	0·473	30	292·6	61·587	0·420
24	291·2	83·856	0·572	30½	291·9	15·948	0·109
24½	294·3	62·567	0·426	31	292	17·481	0·119
25	293	23·198	0·158	31½	293·2	29·034	0·198
25½	290	7·697	0·052	32	291·5	16·972	0·116
26	294·8	23·561	0·161	32½	291	4·020	0·027

These values of η, plotted against the corresponding values of p, give the periodogram shown on p. 357.

In the practice of periodogram analysis, since saving of labour is more important than great accuracy, it is not unusual to omit altogether the calculation of the standard deviations, merely plotting the periodogram from points obtained as follows. As abscissa take p, and as ordinate take the difference between the greatest and least numbers of the sequence $U_0, U_1, U_2, \ldots, U_{p-1}$; this difference is called the *oscillation* corresponding to the trial period p. The table on p. 358 gives the oscillations corresponding to the various values of p, between 19 days and 33½ days, the number m being taken to be 17 throughout.

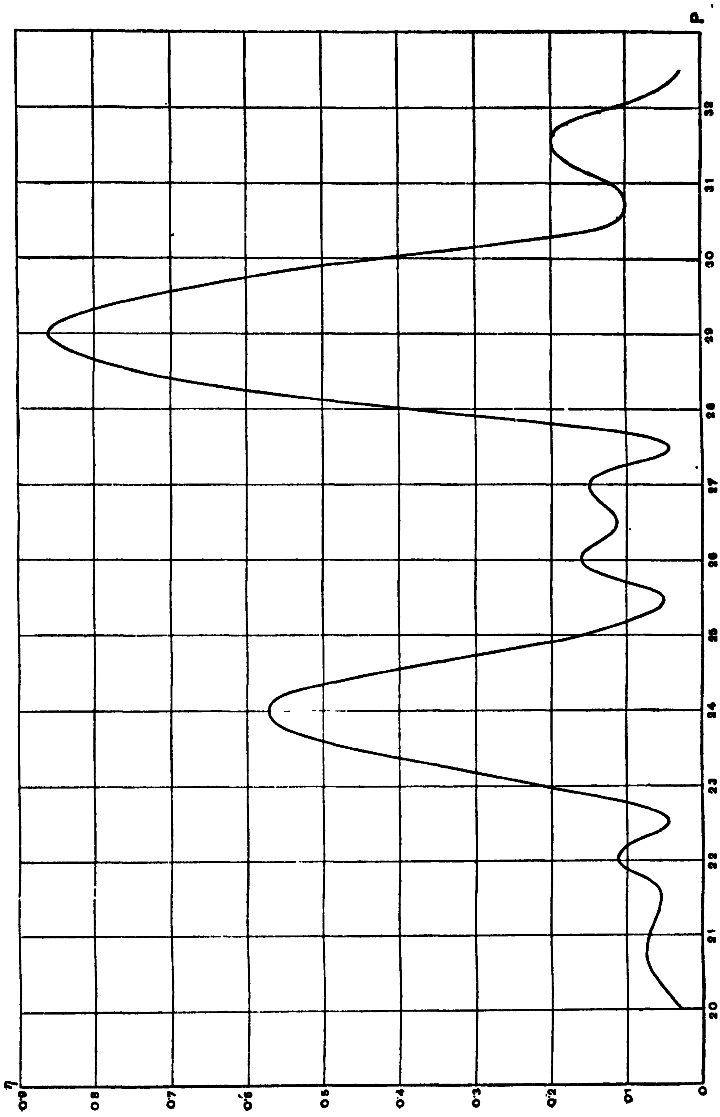

Fig. 19.

Trial Period.	Oscillation of Sum.	Trial Period.	Oscillation of Sum.
Days.		Days.	
19	30	26¼	20
19½	36	26½	50
20	16	27	63
20½	32	27¼	36
21	30	27½	22
21½	27	28	169
22	51	28½	305
22¼	35	29	360
22½	20	29½	303
23	89	30	176
23½	204	30½	55
24	236	31	49
24½	177	31½	86
25	74	32	53
25¼	29	32½	15
25½	23	33	58
26	25	33½	49

It will be seen on plotting the oscillation against p that the curve so obtained closely resembles the periodogram obtained by the more accurate and laborious method of computing the standard deviations.

It will be seen that the periodogram in our example shows two high peaks at $p = 24$ and $p = 29$, with the side-peaks belonging to these as in a diffraction-pattern in optics. We are therefore led to infer the existence of two constituent oscillations, one of which has a period of approximately 24 days and the other of approximately 29 days. In order to find these periods more exactly, we repeat the work so far as concerns the neighbourhood of $p = 24$ and $p = 29$, but taking a larger value of m—say about twice as great—and also taking values of p separated from each other by smaller intervals. Thus we might now calculate the correlation ratios corresponding to $p = 23{\cdot}6$, $23{\cdot}8$, $24{\cdot}0$, $24{\cdot}2$, $24{\cdot}4$, when 34 horizontal lines are taken. This will give a much better defined peak in the neighbourhood of $p = 24$; the peak will in fact be only half as broad as in the previous periodogram.

It may be remarked that if two periods are found (from an inspection of the first periodogram) to be so close together that the peaks corresponding to them run into each other, it will in any case be necessary to repeat the work with a larger value of *m*, in order to diminish the breadth of each peak and so bring the two peaks clear of each other. This is analogous to the corresponding device in spectroscopy, of employing a grating with a larger number of rulings in order to resolve two lines which are not distinctly separated by a smaller instrument.

In order to get the periods still more exactly, we must study the *phase* of the constituent oscillations (as found by Fourier's analysis) at different epochs; for if there is a slight error in the assumed period, the consequence will be that the phases of the oscillation, as determined from different "stretches" of material, will not fit together accurately—the oscillation will appear as if it were continually being accelerated or retarded in phase; an improved value for the period is then suggested by the amount of acceleration or retardation of phase. In this way we find that in the present example the periods are exactly 24 and 29 days. We then add together the two oscillations (*i.e.* write down the numbers M_0, M_1, M_2, . . . for the first oscillation in one horizontal line, and write down the numbers M_0, M_1, M_2, . . . for the second oscillation in a horizontal line below them, and add) and subtract the result from the given observed values, in order to see what is left still unaccounted for. Thus

Day	1	2	3	4	5	6	7	8	9
24-*day term*	16·8	18·8	20·8	21·8	22·9	23·9	23·9	24	23·1
29-*day term*	25·7	26·7	27·8	27·7	27·7	26·6	25·4	24·2	22·1
Sum	42·5	45·5	48·6	49·5	50·6	50·5	49·3	48·2	45·2
Given in graph	25	28	31	32	33	33	32	31	28
Diff.	17·5	17·5	17·6	17·5	17·6	17·5	17·3	17·2	17·2

Day	10	11	12	13	14	15	16	17	18
24-*day term*	22·1	21·1	19·2	17·3	15·2	14·2	12·2	11·2	10·2
29-*day term*	19·9	17·8	15·6	13·5	11·4	9·5	8·4	7·5	6·6
Sum	42·0	38·9	34·8	30·8	26·6	23·7	20·6	18·7	16·8
Given in graph	25	22	18	14	10	7	4	2	0
Diff.	17·0	16·9	16·8	16·8	16·6	16·7	16·6	16·7	16·8

Day	19	20	21	22	23	24	25	26	27
24 *day term*	10·2	10·1	11·1	12	14	14·9	16·8	18·8	20·8
29-*day term*	6·8	6·9	8·1	9·2	11·4	13·6	15·7	17·7	19·8
Sum	17·0	17·0	19·2	21·2	25·4	28·5	32·5	36·5	40·6
Given in graph	0	0	2	4	8	11	15	19	23
Diff.	17·0	17·0	17·2	17·2	17·4	17·5	17·5	17·5	17·6

Day	28	29	30	31	32	33	34	35	36
24-*day term*	21·8	22·9	23·9	23·9	24	23·1	22·1	21·1	19·2
29-*day term*	21·7	23·7	25·7	26·7	27·8	27·7	27·7	26·6	25·4
Sum	43·5	46·6	49·6	50·6	51·8	50·8	49·8	47·7	44·6
Given in graph	26	29	32	33	34	33	32	30	27
Diff.	17·5	17·6	17·6	17·6	17·8	17·8	17·8	17·7	17·6

The numbers in the last line are nearly constant, the deviations from constancy being no more than might be expected from the inaccuracy of the numbers taken to represent the 24-day and 29-day terms; so *the given observations may be accounted for as the resultant of two constituent oscillations of periods* 24 *and* 29 *days respectively, together with a constant term.* We therefore write

$$u_t = a + \beta \cos\frac{2\pi t}{29} + \gamma \sin\frac{2\pi t}{29} + \delta \cos\frac{2\pi t}{24} + \epsilon \sin\frac{2\pi t}{24};$$

the constant a must have the value 17, since this is the mean value of u_t, and the constants $\beta, \gamma, \delta, \epsilon$ may be determined by least squares. The final result is

$$u_t = 17 + 10 \sin\frac{2\pi(t+3)}{29} + 7 \sin\frac{2\pi(t-1)}{24},$$

where t denotes the time in days.

176. Bibliographical Note.—Methods for the discovery of hidden periodicities have been given by many writers, beginning with Lagrange[*] in 1772 and 1778. Lagrange's method, which has been improved by Dale,[†] is quite different from the method described above.

In the latter half of the nineteenth century attention was given chiefly to methods which depend on the principle that (in the notation of § 173) the sequence $U_0, U_1, \ldots, U_{p-1}$ preserves any periodicity of period p that may be present in the observations, and does not preserve other periodicities. Methods of this kind were originated by Buys-Ballot[*] and developed by Strachey[†] and by Stewart and Dodgson.[‡] Stokes[§] suggested that in order to test an observed function $u(x)$ for a period $\frac{2\pi}{n}$, the integrals $\int u(x)\sin nx dx$ and $\int u(x)\cos nx dx$ might be calculated; if a true periodicity of $u(x)$ is represented by the term $c \sin (n'x + a)$, then when n is near n' the integrals will involve terms

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[*] *Œuvres*, 6, p. 505; 7, p. 535.

[†] *Monthly Not. R.A.S.* 74 (1914), p. 628. Carse and Shearer, *Edin. Math. Tracts*, No. 4, p. 41.

$$\frac{c}{2(n'-n)}\sin\{(n'-n)x + a\} \text{ and } -\frac{c}{2(n'-n)}\cos\{(n'-n)x + a\},$$

which are of large amplitude and long period, and are therefore readily detected.

Schuster discussed the matter in a number of important memoirs,[‖] in which the periodogram was introduced. Let a function $u(x)$ of the time x take the values $u_0, u_1, u_2, u_3, \ldots, u_{n-1}$ at equidistant values of the time $x_0, x_0 + a, x_0 + 2a, \ldots, x_0 + (n-1)a$.

Let
$$A = \sum_{s=0}^{n-1} u_s \cos\frac{2\pi s}{p},$$
$$B = \sum_{s=0}^{n-1} u_s \sin\frac{2\pi s}{p},$$
and let
$$S = \frac{(A^2 + B^2)a}{n}.$$

Then the value of S in the neighbourhood of a particular value pa was defined by Schuster to be the ordinate of the periodogram for that period. It was remarked by Craig[¶] that Schuster's formulae were equivalent to those arrived at in finding the correlation coefficient r between the sequence $u_0, u_1, u_2, \ldots, u_{n-1}$ and the sequences

$$1, \quad \cos\frac{2\pi}{p}, \quad \cos\frac{4\pi}{p}, \quad \cos\frac{6\pi}{p}, \ldots.$$

and
$$0, \quad \sin\frac{2\pi}{p}, \quad \sin\frac{4\pi}{p}, \quad \sin\frac{6\pi}{p}, \ldots.$$

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[*] *Les changements périodiques de température*, Utrecht (1847), p. 34.

[†] *Proc. R.S.* **26** (1877), p. 249.

[‡] *Proc. R.S.* **29** (1879), p. 106.

[§] *Proc. R.S.* **29** (1879), pp. 122, 303.

[‖] *Terrestrial Magnetism*, **3** (1898), p. 13; *Camb. Phil. Trans.* **18** (1900), p. 107; *Proc. R.S.* **77** (1906), p. 136; *Phil. Trans.* **206** (1906), p. 69.

[¶] *Brit. Ass. Rep. 1912*, p. 416.

H. H. Turner [*] has published tables for facilitating the computations of Schuster's process.

A method depending on the formation of difference equations has been suggested by Oppenheim,[†] and mechanical methods have been described by A. E. Douglass [‡] and W. L. Balls.[§]

The reader who wishes to pursue further the subject of this chapter is recommended to consult a valuable memoir by J. Bartels, "Random fluctuations, persistence, and quasi-persistence in geophysical and cosmical periodicities", *Terrestrial Mag.* **40** (1935), pp. 1–60, and a memoir by T. E. Sterne and L. Campbell, "Properties of the light curve of ζζ Cygni", *Annals of Harvard Coll. Obs.* **90**, No. 6 (1934). Cf. also Dodd, "Periodogram analysis with the phase a chance variable", *Econometrica* **7** (1939), p. 57; G. T. Walker, "Period-hunting in practice", *Quart. J. R. Met. Soc.* **67** (1941), p. 15; K. Stumpff, *Grundlagen u. Methoden d. Periodenforschung* (Berlin, 1937).

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[*] *Tables for Facilitating the Use of Harmonic Analysis*, by H. H. Turner (Oxford University Press, 1913).

[†] *Wien Sitzungsber.* **118** (2*a*) (1909), p. 823; cf. F. Hopfner, *ibid.* **119** (2*a*) (1910), p. 351.

[‡] *Astrophysical J.* **40** (1914), p. 326; **41** (1915), p. 173.

[§] *Proc. R.S.* **99** (1921), p. 283.

CHAPTER XIV

THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

177. **Theory of the Method.**—The best method of integrating differential equations numerically is one devised by J. C. Adams;* it is applicable to equations of any order, but for simplicity we shall describe its application to equations of the first order.

Let the differential equation be

$$\frac{dy}{dx}=f(y,x), \tag{1}$$

with the initial condition that y is to have the value y_0 when x has the value x_0. Let $x_0, x_1, x_2, x_3, \ldots$ be a sequence of values of x at equal intervals w apart; we shall denote the corresponding values of y by $y_0, y_1, y_2, y_3, \ldots$ and the corresponding values of $w\frac{dy}{dx}$ by $q_0, q_1, q_2, q_3, \ldots$. The differences (y_1-y_0), (y_2-y_1), etc., will be denoted by Δy_0, Δy_1, etc., as usual. It will generally be found convenient to choose the interval w so small that differences of order above the fourth (or, better still, differences of order above the third) may be neglected. The value y_0 being given, the problem is to determine $y_1, y_2, y_3, \ldots$.

The first four of these values are determined in the following way: by differentiating equation (1) we have

$$\frac{d^2y}{dx^2}=\frac{\partial f}{\partial x}+f\frac{\partial f}{\partial y}, \tag{2}$$

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* *Theories of Capillary Action*, by F. Bashforth and J. C. Adams: Cambridge, 1883. The method has been since developed in Russian memoirs by A. Kriloff. C. Störmer, *Comptes rendus du cong. int.*, Strasbourg, 1920, develops a similar method for equations of the second order. For more recent developments cf. R. v. Mises, *Zeitschr. f. angew. Math.* **10** (1930), p. 81; G. Schulz, *ibid.* **12** (1932), p. 44; D. R. Hartree, *Manchester Lit. and Phil. Soc. Mem.* **77** (1932), p. 91; V. M. Falkner, *Phil. Mag.* **21** (1936), p. 624.

which expresses $\frac{d^2y}{dx^2}$ in terms of x and y. By differentiating (2) we can similarly obtain $\frac{d^3y}{dx^3}$ in terms of x and y, and so on. In this way all the successive derivatives of y with respect to x are obtained as known functions of x and y. Thus in Taylor's expansion

$$y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_0 + \frac{(x-x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{(x-x_0)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots \quad (3)$$

the quantities $y_0, \left(\frac{dy}{dx}\right)_0, \left(\frac{d^2y}{dx^2}\right)_0, \ldots$ are all known, and therefore y can be found for any value of x near to x_0. By substituting w, $2w$, $3w$, $4w$ for $(x - x_0)$ in equation (3) we obtain the values of y_1, y_2, y_3, y_4 with as great accuracy as may be desired; or alternatively, we may compute $\Delta y_0, \Delta^2 y_0, \ldots$ from the equations

$$\begin{aligned}
\Delta y_0 &= w\left(\frac{dy}{dx}\right)_0 + \frac{w^2}{2}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{w^3}{6}\left(\frac{d^3y}{dx^3}\right)_0 + \frac{w^4}{24}\left(\frac{d^4y}{dx^4}\right)_0 + \frac{w^5}{120}\left(\frac{d^5y}{dx^5}\right)_0 + \ldots \\
\Delta^2 y_0 &= w^2\left(\frac{d^2y}{dx^2}\right)_0 + w^3\left(\frac{d^3y}{dx^3}\right)_0 + \frac{7}{12}w^4\left(\frac{d^4y}{dx^4}\right)_0 + \frac{1}{4}w^5\left(\frac{d^5y}{dx^5}\right)_0 + \ldots \\
\Delta^3 y_0 &= w^3\left(\frac{d^3y}{dx^3}\right)_0 + \frac{3}{2}w^4\left(\frac{d^4y}{dx^4}\right)_0 + \frac{5}{4}w^5\left(\frac{d^5y}{dx^5}\right)_0 + \ldots \\
\Delta^4 y_0 &= w^4\left(\frac{d^4y}{dx^4}\right)_0 + 2w^5\left(\frac{d^5y}{dx^5}\right)_0 + \ldots \\
\Delta^5 y_0 &= w^5\left(\frac{d^5y}{dx^5}\right)_0 + \ldots
\end{aligned}$$

and then calculate y_1, y_2, y_3, y_4 by building up the difference table of the y's. Then from either the equation

$$q = w\frac{dy}{dx} = wf(y, x)$$

or the equation

$$q = w\left(\frac{dy}{dx}\right)_0 + w(x - x_0)\left(\frac{d^2y}{dx^2}\right)_0 + w\frac{(x-x_0)^2}{2!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$$

we obtain the values of q_1, q_2, q_3, q_4.

Now with these computed values we form a difference table thus:

q.	Δq.	$\Delta^2 q$.	$\Delta^3 q$.	$\Delta^4 q$.
q_0				
	Δq_0			
q_1		$\Delta^2 q_0$		
	Δq_1		$\Delta^3 q_0$	
q_2		$\Delta^2 q_1$		$\Delta^4 q_0$
	Δq_2		$\Delta^3 q_1$	
q_3		$\Delta^2 q_2$		
	Δq_3			
q_4				

The further process consists in extending this table by adjoining new (sloping) lines. To effect this, we remark that to the symbolic formula

$$\mathrm{E}^r = \left(\frac{\mathrm{E}}{\mathrm{E}-\Delta}\right)^r = \left(1-\frac{\Delta}{\mathrm{E}}\right)^{-r} = 1 + \frac{r}{1}\Delta \mathrm{E}^{-1} + \frac{r(r+1)}{1.2}\Delta^2 \mathrm{E}^{-2} + \dots$$

there corresponds the interpolation formula (the backward form of the Gregory-Newton formula)

$$q(x_n + rw) = q_n + \frac{r}{1}\Delta q_{n-1} + \frac{r(r+1)}{1.2}\Delta^2 q_{n-2} + \frac{r(r+1)(r+2)}{1.2.3}\Delta^3 q_{n-3} + \frac{r(r+1)(r+2)(r+3)}{1.2.3.4}\Delta^4 q_{n-4} + \dots \quad (4)$$

Now

$$y_{n+1} - y_n = \int_{x_n}^{x_n+w} \frac{dy}{dx}dx = \frac{1}{w}\int_{x_n}^{x_n+w} q dx = \int_0^1 q(x_n + rw)dr.$$

Substituting the value of $q(x_n + rw)$ from (4) and performing the integrations with respect to r, we have

$$y_{n+1} - y_n = q_n + \frac{1}{2}\Delta q_{n-1} + \frac{5}{12}\Delta^2 q_{n-2} + \frac{3}{8}\Delta^3 q_{n-3} + \frac{251}{720}\Delta^4 q_{n-4} + \dots \quad (5)$$

and in particular

$$y_5 - y_4 = q_4 + \frac{1}{2}\Delta q_3 + \frac{5}{12}\Delta^2 q_2 + \frac{3}{8}\Delta^3 q_1 + \frac{251}{720}\Delta^4 q_0.$$

Every term on the right-hand side of this equation is known, so by means of it we can compute y_5. The equation

$$q_5 = wf(y_5, x_5)$$

enables us now to compute q_5, and so to adjoin a new sloping line to the difference table of the q's.

Next we put $n = 5$ in equation (5) and use this equation to compute y_6; then from the equation

$$q_6 = wf(y_6, x_6)$$

we compute q_6, and so obtain another sloping line in the difference table.

Adams's process consists simply in the repetition of this operation. It may evidently be extended to any number of simultaneous equations each of the first order, such as the pair

$$\frac{dy}{dx} = \phi(y, z, x),$$

$$\frac{dz}{dx} = \psi(y, z, x),$$

and so to any system of ordinary differential equations.

Ex.—Given the differential equation

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

with the initial values $x_0 = 0$, $y_0 = 1$, *tabulate the solution from* $x = 0$ *to* $x = 0{\cdot}2$.

We have by successive differentiation

$$\begin{aligned} yy' + xy' - y + x &= 0, \\ y'^2 + yy'' + xy'' + 1 &= 0, \\ 3y'y'' + yy''' + y'' + xy''' &= 0, \\ 3y''^2 + 4y'y''' + yy^{\text{IV}} + xy^{\text{IV}} + 2y''' &= 0, \\ 10y''y''' + 5y'y^{\text{IV}} + yy^{\text{V}} + xy^{\text{V}} + 3y^{\text{IV}} &= 0, \\ 10y'''^2 + 15y''y^{\text{IV}} + 6y'y^{\text{V}} + yy^{\text{VI}} + xy^{\text{VI}} + 4y^{\text{V}} &= 0, \end{aligned}$$

whence the initial values are

$$y_0' = 1, \quad y_0'' = -2, \quad y_0''' = 8, \quad y_0^{\text{IV}} = -60, \quad y_0^{\text{V}} = 640, \quad y_0^{\text{VI}} = -8840,$$

and hence the Taylor series is

$$y = 1 + x - x^2 + \frac{4}{3}x^3 - \frac{5}{2}x^4 + \frac{16}{3}x^5 - \frac{221}{18}x^6 + \ldots$$

with

$$y' = 1 - 2x + 4x^2 - 10x^3 + \frac{80}{3}x^4 - \frac{221}{3}x^5 + \ldots$$

Taking $w = 0{\cdot}02$, we have from these series

$y_0 = 1{\cdot}000000$, $q_0 = 0{\cdot}02$,
$y_1 = 1{\cdot}019610$, $q_1 = 0{\cdot}019230$,
$y_2 = 1{\cdot}038479$, $q_2 = 0{\cdot}018516$,
$y_3 = 1{\cdot}056659$, $q_3 = 0{\cdot}017851$,
$y_4 = 1{\cdot}074195$. $q_4 = 0{\cdot}017228$.

The remaining values $y_5, y_6, y_7, \ldots$ are computed from a difference table of the q's, using equation (5); the corresponding values $q_5, q_6, q_7, \ldots$ being given by the formula $q_n = w(y_n - x_n)/(y_n + x_n)$. We thus arrive at the following results:

			Δ.	Δ^2.	Δ^3.	Δ^4.
$x_0 = 0$	$y_0 = 1{\cdot}000000$	$q_0 = 0{\cdot}02$				
			-770			
$x_1 = 0{\cdot}02$	$y_1 = 1{\cdot}019610$	$q_1 = 0{\cdot}019230$		56		
			-714		-7	
$x_2 = 0{\cdot}04$	$y_2 = 1{\cdot}038479$	$q_2 = 0{\cdot}018516$		49		0
			-665		-7	
$x_3 = 0{\cdot}06$	$y_3 = 1{\cdot}056659$	$q_3 = 0{\cdot}017851$		42		2
			-623		-5	
$x_4 = 0{\cdot}08$	$y_4 = 1{\cdot}074195$	$q_4 = 0{\cdot}017228$		37		2
			-586		-3	
$x_5 = 0{\cdot}10$	$y_5 = 1{\cdot}091126$	$q_5 = 0{\cdot}016642$		34		-2
			-552		-5	
$x_6 = 0{\cdot}12$	$y_6 = 1{\cdot}107490$	$q_6 = 0{\cdot}016090$		29		4
			-523		-1	
$x_7 = 0{\cdot}14$	$y_7 = 1{\cdot}123317$	$q_7 = 0{\cdot}015567$		28		-3
			-495		-4	
$x_8 = 0{\cdot}16$	$y_8 = 1{\cdot}138632$	$q_8 = 0{\cdot}015072$		24		
			-471			
$x_9 = 0{\cdot}18$	$y_9 = 1{\cdot}153469$	$q_9 = 0{\cdot}014601$				
$x_{10} = 0{\cdot}20$	$y_{10} = 1{\cdot}167842$					

The above value of y_{10} is correct to the last digit.

178. **Bibliographical Note.**—Of the other methods which have been proposed for integrating differential equations, the best known is that of Runge, *Math. Ann.* **46** (1895), p. 167, improved and extended by Kutta, *Zeits. f. Math. u. Phys.* **46** (1901), p. 435. See also Lindelöf, *Acta Soc. Sc. Fenn.* **2** (1938), No. 13.

A method which permits the determination of an upper limit to the error involved has been described by Steffensen, *Särtryck ur Skandinavisk Aktuarietidskrift*, 1922, p. 20.

On the numerical solution of partial differential equations, cf. Gorakh Prasad, *Phil. Mag.* **9** (1930), 1074. A valuable monograph on the numerical integration of differential equations, both ordinary and partial, written by A. A. Bennett, W. E. Milne and H. Bateman, was published by the National Research Council, Washington, D.C., in 1933.

CHAPTER XV

SOME FURTHER PROBLEMS

IN this chapter we shall give a brief treatment of various topics which cannot be discussed more fully here on account of limitations of space.

179. **The Summation of Slowly - Convergent Series.**—Many of the commonest series of Analysis converge very slowly. Thus with Brouncker's series for $\log_e 2$,

$$\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \ldots$$

a hundred terms are required to give the sum accurately to two digits; 10,000 terms are required to give it accurately to four digits; and 1,000,000,000 terms are required to give it accurately to nine digits.

Stirling,* in 1730, showed how a series of this kind may be transformed into one which is rapidly convergent.

His method is to expand the general term of the given series as a series of inverse factorials; thus the general term of Brouncker's series is $u_x = \frac{1}{4x(x+\frac{1}{2})}$, where $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and this may be expanded as a series of inverse factorials in the form

$$u_x = \frac{1}{2^2x(x+1)} + \frac{1}{2^3x(x+1)(x+2)} + \frac{1.3}{2^4x(x+1)(x+2)(x+3)} + \frac{1.3.5}{2^5x(x+1)(x+2)(x+3)(x+4)} + \ldots$$

Now form the sum $u_x + u_{x+1} + u_{x+2} + u_{x+3} + \ldots$ The sums arising from the individual inverse factorials can each be summed by the well-known algebraical formula, and thus we obtain

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* *Meth. Diff.* (1730), Prop. II. Ex. 5.

$$u_x + u_{x+1} + u_{x+2} + \ldots = \frac{1}{2^2 x} + \frac{\frac{1}{2}}{2^3 x(x+1)} + \frac{\frac{1}{3}.1.3}{2^4 x(x+1)(x+2)} + \frac{\frac{1}{4}.1.3.5}{2^5 x(x+1)(x+2)(x+3)} + \ldots$$

In this formula put $x = 13\frac{1}{2}$. Thus

$$\frac{1}{27.28} + \frac{1}{29.30} + \frac{1}{31.32} + \ldots = \frac{1}{2^2.\frac{27}{2}} + \frac{\frac{1}{2}}{2^3.\frac{27}{2}.\frac{29}{2}} + \frac{\frac{1}{3}.1.3}{2^4.\frac{27}{2}.\frac{29}{2}.\frac{31}{2}} + \frac{\frac{1}{4}.3.5}{2^5.\frac{27}{2}.\frac{29}{2}.\frac{31}{2}.\frac{33}{2}} + \ldots$$

The series on the right is rapidly convergent and yields the sum 0·018861219 . . .; and the sum of the first thirteen terms of Brouncker's series $\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{25.26}$ is found by addition to be 0·674285961 . . . Adding these sums, we have finally for the sum to infinity of Brouncker's series

$$\log_e 2 = 0{\cdot}693147180.$$

Stirling's method was extended by himself (*Meth. Diff.* (1730), Prop. III.) to the case of series whose *n*th term involves an *n*th power. Another method was given by Kummer in *Journal für Math.* **16** (1837), p. 206. See further Catalan, *Mém. Belg. cour.* **33** (1865); Markoff, *Mém. de St-Pét.* (7) **37** (1890); Andoyer, *Bull. de la Soc. Math. de France*, **33** (1905), p. 36; and Bockwinkel, *Nieuw Archief voor Wiskunde* (2) **13** (1921), p. 383.

180. **Prony's Method of Interpolation by Exponentials.***— We shall now show how a function $\kappa(x)$, which is specified by a table of numerical values, may be represented approximately by a sum of exponentials

$$\kappa(x) = Pe^{px} + Qe^{qx} + Re^{rx} + \ldots + Ve^{vx},$$

where P, Q, R, . . ., V, $p, q, r, \ldots, v$ are constants which are chosen so as to give the closest possible representation of the given numerical values. Let the given values of $\kappa(x)$ be $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots$ corresponding respectively to the values $0, w, 2w, 3w, \ldots$ of the argument x.

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* A. L. Prony, *Jour. de l'Éc. Pol.* Cah. 2 (an IV.), p. 29.

If $\kappa(x)$ could be represented *exactly* as a sum of μ exponentials, say

$$Pe^{px} + Qe^{qx} + Re^{rx} + \ldots + Ve^{vx},$$

then $\kappa(x)$ would satisfy a linear difference equation of the form

$$A\kappa_{n+\mu} + B\kappa_{n+\mu-1} + C\kappa_{n+\mu-2} + \ldots + M\kappa_n = 0,$$

where the roots of the algebraic equation

$$Ax^{\mu} + Bx^{\mu-1} + \ldots + M = 0$$

would be $e^{pw}, e^{qw}, \ldots, e^{vw}$. Prony's method, which is based on this fact, is to write down a set of linear equations

$$\begin{array}{l} A\kappa_{\mu} \quad + B\kappa_{\mu-1} + C\kappa_{\mu-2} + \ldots + M\kappa_0 = 0, \\ A\kappa_{\mu+1} + B\kappa_{\mu} \quad + C\kappa_{\mu-1} + \ldots + M\kappa_1 = 0, \\ A\kappa_{\mu+2} + B\kappa_{\mu+1} + C\kappa_{\mu} \quad + \ldots + M\kappa_2 = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

(where the quantities $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots$ are known, since $\kappa(x)$ is a known tabulated function), and by the ordinary method of Least Squares to find the values of A, B, C, . . ., M which best satisfy these equations; then with these values of A, B, C, . . ., M to form the algebraic equation

$$Ax^{\mu} + Bx^{\mu-1} + \ldots + M = 0$$

and find its roots; these roots will be $e^{pw}, e^{qw}, \ldots, e^{vw}$, and thus $p, q, \ldots, v$ are determined. Knowing $p, q, \ldots, v$, we have a set of linear equations to determine the coefficients P, Q, . . ., V, and these also are to be solved by the method of Least Squares.

Ex. 1.—*If from the data*

x	10	20	30	40	50	60	70	80	90
$\kappa(x)$	6460	6090	5642	5049	4417	3623	2401	983	142

we represent $\kappa(x)$ *in the form*

$$P\alpha^x + Q\beta^x + R\gamma^x,$$

prove that the best values of α, β, γ *are the roots of the cubic*

$$x^3 - 3{\cdot}02923x^2 + 3{\cdot}78779x - 1{\cdot}68664 = 0.$$

(W. S. B. Woolhouse.)

Ex. 2.—*From the data*

x	0	8	16	24	32	40
$\kappa(x)$	248	345	421	481	529	569

show that

$$\begin{aligned}\kappa(x) = {} & 629{\cdot}3763 \times 1{\cdot}0014438^x \\ & - 381{\cdot}3767 \times 0{\cdot}9670412^x \\ & + 0{\cdot}00046064 \times 1{\cdot}2361997^x\end{aligned}$$

(F. Selling.)

181. **Interpolation Formulae for Functions of Two Arguments.***—The formulae of interpolation for functions of a single argument, which have been established in Chapters I.-III., may be extended to functions of two or more independent arguments. Thus we can introduce *divided differences* (cf. § 11) for a function $f(x, y)$ of two arguments by the definition

$$\begin{aligned}f_{\kappa,\lambda}(x, y) &= \frac{f_{\kappa,\lambda-1}(x, y) - f_{\kappa,\lambda-1}(x, y_\lambda)}{y - y_\lambda} \\ &= \frac{f_{\kappa-1,\lambda}(x, y) - f_{\kappa-1,\lambda}(x_\kappa, y)}{x - x_\kappa},\end{aligned}$$

and obtain a formula analogous to Newton's formula for unequal intervals (§ 13), namely,

$$\begin{aligned}f(x, y) &= f_{00}(x, y) \\ &= f_{00}(x_1, y_1) + (x - x_1)f_{10}(x, y) + (y - y_1)f_{01}(x_1, y) \\ &= f_{00}(x_1, y_1) + (x - x_1)f_{10}(x, y_1) + (y - y_1)f_{01}(x, y) \\ &= f_{00}(x_1, y_1) + (x - x_1)f_{10}(x_2, y_1) + (x - x_1)(x - x_2)f_{20}(x, y_1) \\ &\qquad + (y - y_1)f_{01}(x_1, y_2) + (x - x_1)(y - y_1)f_{11}(x, y) \\ &\qquad\qquad + (y - y_1)(y - y_2)f_{02}(x_1, y),\end{aligned}$$

etc.

From this all the interpolation formulae for a function of two arguments may be derived. Thus the polynomial of the second degree, which at the places

$$\begin{array}{lll}(x_1, y_1) & (x_1, y_2) & (x_1, y_3) \\ (x_2, y_1) & (x_2, y_2) & \\ (x_3, y_1) & & \end{array}$$

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* Cf. K. Pearson, *Tracts for Computers*, No. III. (Cambridge, 1920); completed (by the discussion of some omitted cases) on pp. x *et seq.* of the Introduction to *Tables of the Incomplete Gamma Function*; S. Narumi, *Tôhoku Math. Journ.* 18 (1920), p. 309.

takes the values

$$\begin{array}{lll} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \\ \eta_{31} & & \end{array}$$

respectively, is

$$\begin{aligned} f(x, y) = \eta_{11} &+ \left(\frac{\eta_{11}}{x_1 - x_2} + \frac{\eta_{21}}{x_2 - x_1}\right)(x - x_1) + \left(\frac{\eta_{11}}{y_1 - y_2} + \frac{\eta_{12}}{y_2 - y_1}\right)(y - y_1) \\ &+ \left(\frac{\eta_{11}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\eta_{21}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\eta_{31}}{(x_3 - x_1)(x_3 - x_2)}\right) \\ &\qquad . (x - x_1)(x - x_2) \\ &+ \left(\frac{\eta_{11}}{(x_1 - x_2)(y_1 - y_2)} + \frac{\eta_{12}}{(x_1 - x_2)(y_2 - y_1)} + \frac{\eta_{21}}{(x_2 - x_1)(y_1 - y_2)}\right. \\ &\qquad \left. + \frac{\eta_{22}}{(x_2 - x_1)(y_2 - y_1)}\right)(x - x_1)(y - y_1) \\ &+ \left(\frac{\eta_{11}}{(y_1 - y_2)(y_1 - y_3)} + \frac{\eta_{12}}{(y_2 - y_1)(y_2 - y_3)} + \frac{\eta_{13}}{(y_3 - y_1)(y_3 - y_2)}\right) \\ &\qquad . (y - y_1)(y - y_2). \end{aligned}$$

In particular, taking $x_1 = y_1 = 0$, $x_2 = y_2 = 1$, $x_3 = y_3 = -1$, we have the formula

$$\begin{aligned} f(x, y) = f_{0,0} &+ \tfrac{1}{2}x(f_{1,0} - f_{-1,0}) + \tfrac{1}{2}y(f_{0,1} - f_{0,-1}) \\ &+ \tfrac{1}{2}x^2(f_{1,0} - 2f_{0,0} + f_{-1,0}) + xy(f_{0,0} - f_{0,1} - f_{1,0} + f_{1,1}) \\ &+ \tfrac{1}{2}y^2(f_{0,1} - 2f_{0,0} + f_{0,-1}), \end{aligned}$$

which is the best for general use when x and y are positive and less than $\frac{1}{2}$.

Similarly we may determine* the polynomial of degree m which at the $\frac{1}{2}(m+1)(m+2)$ places

$$\begin{array}{llllll} (x_1, y_1) & (x_2, y_1) & \cdots & (x_m, y_1) & (x_{m+1}, y_1) \\ (x_1, y_2) & (x_2, y_2) & \cdots & (x_m, y_2) & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (x_1, y_{m+1}) & & & & \end{array}$$

takes the values

$$\begin{array}{lllll} \eta_{11} & \eta_{21} & \cdots & \eta_{m1} & \eta_{m+1,1} \\ \eta_{12} & \eta_{22} & \cdots & \eta_{m2} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \eta_{1,m+1}. & & & & \end{array}$$

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* Cf. O. Biermann, *Monatshefte für Math.* **14** (1903), p. 211.

Considering now the case when the entries are given at equal intervals ξ of x and equal intervals η of y, and introducing the notation

$$\begin{aligned}
\Delta_\xi f &= f(x+\xi,\, y) - f(x,\, y),\\
\Delta_\eta f &= f(x,\, y+\eta) - f(x,\, y),\\
\Delta^2_{\xi\eta} f &= f(x+\xi,\, y+\eta) - f(x,\, y+\eta) - f(x+\xi,\, y) + f(x,\, y),\\
&\text{etc.,}
\end{aligned}$$

the above generalisation of Newton's formula for unequal intervals becomes

$$\begin{aligned}
f(x+m\xi,\, y+n\eta)
&= f(x,\, y) + m\Delta_\xi f + n\Delta_\eta f + \frac{m(m-1)}{2}\Delta^2_{\xi\xi} f + mn\Delta^2_{\xi\eta} f\\
&\quad + \frac{n(n-1)}{2}\Delta^2_{\eta\eta} f + \frac{m(m-1)(m-2)}{2.3}\Delta^3_{\xi\xi\xi} f + \frac{m(m-1)}{2} n\Delta^3_{\xi\xi\eta} f\\
&\quad + \frac{mn(n-1)}{2}\Delta^3_{\xi\eta\eta} f + \frac{n(n-1)(n-2)}{2.3}\Delta^3_{\eta\eta\eta} f + \ldots
\end{aligned}$$

This, which may be regarded as the extension of the Gregory-Newton formula to functions of two arguments, is due to Lambert.* Symbolically it may be written

$$f(x+m\xi, y+n\eta) = (1+\Delta_\xi)^m (1+\Delta_\eta)^n f(x, y).$$

It is sometimes practicable and advantageous to reduce interpolation of functions of two arguments to linear interpolation.† For example, if $u(x, y)$ is tabulated for quinary values of x and y, the u corresponding to any integral values of x and y may, by a proper choice of origin, be reduced to the form $u(\pm t, \pm s)$, where t and s take one of the values 1 and 2. The value of u can be found by interpolation along the line of values of the form $u(\pm 5t, \pm 5s)$: for example, $u(-1, 2)$ may be found by linear interpolation along the line of values . . . $u(-5, 10)$, $u(0, 0)$, $u(5, -10)$ . . .

Ex. 1.—*Complete the accompanying table on the assumption that third-order differences are everywhere zero, and express the tabulated function as a polynomial in two variables.*

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* *Beyträge*, Part III. Cf. Lagrange, *Nouv. Mém. de Berlin* (1772), reprinted *Œuvres*, **3**, p. 441.

† Elderton, *Biometrika*, **6** (1908), p. 94; Spencer, *J.I.A.* **40**, p. 299; Burn and Brown, *Elements of Finite Differences*, §§ 148-154.

	$x=0$	1	2	3
$y=0$	3	5	9	
1	2	3		
2	3			
3				

[*Answer*, $u=3+x-2y+x^2-xy+y^2$.]

Ex. 2.—*The following table gives the time (in hours, minutes, and seconds) corresponding to certain altitudes* (a) *of the sun in various declinations* (δ) *at a place in a certain latitude.*

	$a=10°$			14°			18°			22°		
$\delta=20°$	6^h	11^m	26^s	5^h	50^m	17^s	5^h	29^m	27^s	5^h	8^m	48^s
15°	5	55	41	5	35	5	5	14	39	4	54	17
10°	5	40	16	5	19	56	4	59	37	4	39	17
5°	5	24	50	5	4	30	4	44	4	4	23	29
0°	5	9	5	4	48	29	4	27	39	4	6	28

Find the time corresponding to $a=16°$, $\delta=12°$.

[*Answer*, $5^h\ 15^m\ 50^s$.]

Find also the time corresponding to $a=20°$, $\delta=14°$.

[*Answer*, $5^h\ 1^m\ 30^s$.]

Ex. 3.—The construction of isobars on meteorological charts essentially involves inverse interpolation with two arguments. The ordinary construction assumes that, within small intervals, the barometric pressure is a linear function of the co-ordinates of the place. A more accurate construction due to Thiele [*Tidsskr.*(3) 4 (1873), p. 87], which is specially useful near maxima and minima, is this. Three neighbouring places of observation are connected by lines, and on these the points are found where, according to the ordinary method, there would be a certain barometric pressure u. The line joining any two of these points cuts the circumscribing circle of the triangle in points which lie on the isobar belonging to the pressure u.

Show that Thiele's construction may be derived from the assumption that the barometric pressure may be expressed by a function of the form

$$u=a+bx+cy+d(x^2+y^2).$$

182. The Numerical Computation of Double Integrals.—It is easy to construct formulae for the evaluation of double integrals on the same principle as the Newton-Cotes and Gauss formulae of single integration (§§ 76, 80). Thus when differences of the fourth order (in the two variables combined) are neglected, we have (by a double application of Simpson's formula)

$$\int_c^d dy \int_a^b f(x, y)dx = \tfrac{1}{36}(b-a)(d-c)\Big\{f(a, c) + f(a, d) + f(b, c)$$
$$+ f(b, d) + 4\Big[f\Big(a, \frac{c+d}{2}\Big) + f\Big(b, \frac{c+d}{2}\Big) + f\Big(\frac{a+b}{2}, c\Big) + f\Big(\frac{a+b}{2}, d\Big)\Big]$$
$$+ 16f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big)\Big\}.$$

This formula indeed is true even when some of the differences of orders 4 to 9 are *not* negligible, *e.g.* $\Delta_x\Delta_y^3$, $\Delta_x^2\Delta_y^2$, $\Delta_x^3\Delta_y$, $\Delta_x^2\Delta_y^3$, $\Delta_x^3\Delta_y^2$, $\Delta_x^3\Delta_y^3$.

When differences of the sixth order are neglected, we have Burnside's formula,*

$$\int_{-1}^1 \int_{-1}^1 f(x, y)dxdy = \frac{40}{49}\Big\{f\Big(\sqrt{\frac{7}{15}}, 0\Big) + f\Big(0, \sqrt{\frac{7}{15}}\Big)$$
$$+ f\Big(-\sqrt{\frac{7}{15}}, 0\Big) + f\Big(0, -\sqrt{\frac{7}{15}}\Big)\Big\} + \frac{9}{49}\Big\{f\Big(\sqrt{\frac{7}{9}}, \sqrt{\frac{7}{9}}\Big)$$
$$+ f\Big(\sqrt{\frac{7}{9}}, -\sqrt{\frac{7}{9}}\Big) + f\Big(-\sqrt{\frac{7}{9}}, \sqrt{\frac{7}{9}}\Big) + f\Big(-\sqrt{\frac{7}{9}}, -\sqrt{\frac{7}{9}}\Big)\Big\}.$$

This is exact so long as $f(x, y)$ is a polynomial of degree not exceeding 5.

A formula similar to Burnside's may be obtained by a double application of Gauss's three-term (fifth-difference) formula. This also is valid even when many of the higher differences are not negligible.

It is often advantageous to break up the field of the double integration into sections, and apply a formula to each section separately.

For other formulae cf. W. F. Sheppard, *Proc. Lond. Math. Soc.* **31** (1899), p. 486; **32** (1900), p. 272; A. C. Aitken and G. L. Frewin, *Proc. Edin. Math. Soc.* **42** (1923-4); M. Sadowski, *Amer. Math. Monthly* **47** (1940).

Ex. 1.—*Show that the value of* $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(3-x^2-y^2)}}$ *computed by Burnside's formula, is* 0 6641.

[*The true value is* 0·6638 *to* 4 *digits.*]

Ex. 2.—*Show that the value of* $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(2-x^2-y^2)}}$ *computed by Burnside's formula, is* 0·9262.

[*The true value is* 0·9202 *to* 4 *digits.*]

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* *Mess. of Maths.* (2) **37** (1908), p. 166.

183. The Numerical Solution of Integral Equations.—In recent years Integral Equations have proved to be of great importance in Applied Mathematics; thus it was by the numerical solution of an integral equation that Knott was enabled in 1919 [*] to deduce the forms of the seismic rays in the earth's interior by a rigorous mathematical method from the observational data of earthquakes.

Some types of integral equation which occur in Applied Mathematics are as follows:

(i.) *Abel's Original Equation.*—This is

$$\int_0^x \frac{\phi(s)ds}{(x-s)^p} = f(x), \qquad (0<p<1, f(0)=0) \tag{1}$$

where $\phi(x)$ is the unknown function which is to be determined, and $f(x)$ is a given function. *The solution of this equation,* which was given by Abel himself,[†] *is*

$$\phi(x) = \frac{1}{\pi}\sin p\pi \int_0^x \frac{f'(s)ds}{(x-s)^{1-p}}. \tag{2}$$

When $f(x)$ is given, the values of $f'(s)$ and of this integral may be obtained without difficulty by the ordinary processes of interpolation and numerical integration.

(ii.) *Integral Equations of Abel's Type.*—These, which may be regarded as a generalisation of (1), are of the form

$$\int_0^x \phi(s)\kappa(x-s)ds = f(x), \tag{3}$$

where $\kappa(x)$ is a given function called the *nucleus,* $f(x)$ is also a given function, and $\phi(x)$ is the unknown function which is to be determined. We need only consider the case when the nucleus

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[*] *Proc. R.S.E.* 39 (1919), p. 157. It was H. Bateman, *Phil. Mag.* (6), 19 (1910), p. 576, who discovered the integral equation.

[†] *Œuvres* (ed. 1881), p. 11 (1823) and p. 97 (1826). The fundamental meaning of Abel's result is most clearly seen if the integrals which occur in it are interpreted as in the theory of generalised differentiation; if $\psi(x)$ is written for $\Gamma(1-p)\phi(x)$, Abel's formula reduces to the simple statement that if

$$\left(\frac{d}{dx}\right)^{p-1}\psi(x) = f(x),$$

then

$$\psi(x) = \left(\frac{d}{dx}\right)^{-p} f'(x).$$

$\kappa(x)$ becomes infinite at $x=0$, for in the simpler case, when the nucleus is finite at $x=0$, the equation can be reduced immediately (by differentiating it) to Poisson's type (iii.) and dealt with by the methods appropriate to that type. We may then suppose $\kappa(x)$ to be such that $x^p\kappa(x)$ is finite and not zero at $x=0$, where p lies between 0 and 1; and for the purposes of numerical integration we can represent it, by the methods of Chapters I.-III., with as great a degree of accuracy as may be desired, by an analytical expression of the form

$$\kappa(x)=x^{-p}(a_0+a_1x+a_2x^2+\ .\ .\ .+a_nx^n).$$

Then* *the solution of the integral equation* (3) *is*

$$\phi(x)=\frac{\sin p\pi}{\pi}\int_0^x f'(s)\mathrm{L}(x-s)ds,$$

where

$$\mathrm{L}(x)=\frac{x^{p-1}}{a_0}+\frac{\alpha^{n-p}}{\mathrm{F}'(\alpha)}\gamma_p(\alpha x)+\frac{\beta^{n-p}}{\mathrm{F}'(\beta)}\gamma_p(\beta x)+\ .\ .\ .+\frac{\nu^{n-p}}{\mathrm{F}'(\nu)}\gamma_p(\nu x),$$

and where $\alpha, \beta, \ .\ .\ .,\ \nu$ *are the roots of the algebraic equation*

$$\begin{aligned}\mathrm{F}(x)\equiv a_0x^n+(1-p)a_1x^{n-1}+(1-p)(2-p)a_2x^{n-2}+\ .\ .\ .\\ +(1-p)(2-p)\ .\ .\ .\ (n-p)a_n=0,\end{aligned}$$

and where $\gamma_p(x)$ *denotes the Incomplete Gamma Function*

$$\gamma_p(x)=e^x\int_0^x s^{p-1}e^{-s}ds.$$

This may be regarded as a direct extension of Abel's original formula (2), which may be derived from it by taking $n=0$. It expresses the solution of the integral equation in a finite form in terms of the Incomplete Gamma Function, of which tables have been published.†

(iii.) *Integral Equations of Poisson's Type.*—These are of the form

$$\phi(x)+\int_0^x\phi(s)\kappa(x-s)ds=f(x),\qquad(4)$$

where $\kappa(x)$ and $f(x)$ are given functions, and $\phi(x)$ is the unknown function which is to be determined. Several different methods for the numerical integration of this equation are given in

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* Whittaker, *Proc. R.S.* **94** (1918), p. 367.

† K. Pearson, *Tables of the Incomplete Gamma Function* (1922).

Whittaker's memoir of 1918,[*] and have been applied to the solution of certain problems in viscous fluid motion by Havelock.[†] We shall here indicate the most useful of them.

Since the nucleus $\kappa(x)$ is supposed to be specified by a table of numerical values over the range of values of x considered, we may apply Prony's method of interpolation by exponentials in order to represent it analytically in the form of a sum of μ exponentials

$$\kappa(x) = \mathrm{P}e^{px} + \mathrm{Q}e^{qx} + \ldots + \mathrm{V}e^{vx}, \tag{5}$$

where ($\mathrm{P}, \mathrm{Q}, \ldots, \mathrm{V}, p, q, \ldots, v$) are constants which are chosen so as to give the closest possible representation of the given numerical values. Taking then this form (5) for the nucleus $\kappa(x)$, we shall show that the integral equation (4) may be satisfied by a solution of the form

$$\phi(x) = f(x) - \int_0^x \mathrm{K}(x-s)f(s)ds, \tag{6}$$

where the solving function $\mathrm{K}(x)$ is also a sum of μ exponentials, say

$$\mathrm{K}(x) = \mathrm{A}e^{\alpha x} + \mathrm{B}e^{\beta x} + \mathrm{C}e^{\gamma x} + \ldots + \mathrm{N}e^{\nu x}. \tag{7}$$

To prove this we remark first that certain existence-theorems established by Volterra justify us in assuming for the solution the form (6), where $\mathrm{K}(x)$ is now the function to be determined. In (6) put $\kappa(x)$ for $f(x)$: thus

$$\phi(x) = \kappa(x) - \int_0^x \mathrm{K}(x-s)\kappa(s)ds,$$

which gives the value of $\phi(x)$ corresponding to this value of $f(x)$.

Putting $(x-s)$ for s in the integral, we have

$$\phi(x) = \kappa(x) - \int_0^x \mathrm{K}(s)\kappa(x-s)ds.$$

Comparing this with the integral equation (4), after replacing $f(x)$ by $\kappa(x)$ in the latter, we have

$$\phi(x) = \mathrm{K}(x),$$

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[*] *Loc. cit.*

[†] *Phil. Mag.* **42** (1921), pp. 620, 628.

and therefore the pair of functions

$$\phi(x)=\mathrm{K}(x),\quad f(x)=\kappa(x)$$

satisfy the integral equation; that is to say,

$$\mathrm{K}(x)+\int_0^x \mathrm{K}(s)\kappa(x-s)ds=\kappa(x). \tag{8}$$

In this equation substitute the value (5) for $\kappa(x)$ and the value (7) for $\mathrm{K}(x)$. Thus we have

$$\begin{aligned}&\mathrm{A}e^{\alpha x}+\mathrm{B}e^{\beta x}+\mathrm{C}e^{\gamma x}+\ldots+\mathrm{N}e^{\nu x}\\&\quad+\int_0^x\{\mathrm{A}e^{\alpha s}+\mathrm{B}e^{\beta s}+\mathrm{C}e^{\gamma s}+\ldots+\mathrm{N}e^{\nu s}\}\\&\qquad\qquad\{\mathrm{P}e^{px-ps}+\mathrm{Q}e^{qx-qs}+\ldots+\mathrm{V}e^{vx-vs}\}ds\\&\quad=\mathrm{P}e^{px}+\mathrm{Q}e^{qx}+\ldots+\mathrm{V}e^{vx}.\end{aligned}$$

Equating coefficients of $e^{\alpha x}$ on the two sides of the equation, we have

$$\frac{\mathrm{P}}{\alpha-p}+\frac{\mathrm{Q}}{\alpha-q}+\ldots+\frac{\mathrm{V}}{\alpha-v}+1=0,$$

and similarly equating coefficients of $e^{\beta x}$

$$\frac{\mathrm{P}}{\beta-p}+\frac{\mathrm{Q}}{\beta-q}+\ldots+\frac{\mathrm{V}}{\beta-v}+1=0,$$

and so on; and therefore $\alpha, \beta, \gamma, \ldots, \nu$ are the roots of the algebraic equation in x,

$$\frac{\mathrm{P}}{x-p}+\frac{\mathrm{Q}}{x-q}+\ldots+\frac{\mathrm{V}}{x-v}+1=0. \tag{9}$$

This enables us to determine $\alpha, \beta, \gamma, \ldots$.

Next, equating coefficients of e^{px} on the two sides of the equation, we have

Similarly

$$\left.\begin{aligned}&\frac{\mathrm{A}}{\alpha-p}+\frac{\mathrm{B}}{\beta-p}+\ldots+\frac{\mathrm{N}}{\nu-p}+1=0\\&\frac{\mathrm{A}}{\alpha-q}+\frac{\mathrm{B}}{\beta-q}+\ldots+\frac{\mathrm{N}}{\nu-q}+1=0\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\&\frac{\mathrm{A}}{\alpha-v}+\frac{\mathrm{B}}{\beta-v}+\ldots+\frac{\mathrm{N}}{\nu-v}+1=0\end{aligned}\right\}. \tag{10}$$

Since $(\alpha, \beta, \gamma, \ldots, \nu)$ and $(p, q, r, \ldots, v)$ are known, these equations (10) enable us to determine A, B, . . ., N; and we see that if the constants $(\alpha, \beta, \gamma, \ldots, \nu)$ and (A, B, . . ., N) are determined by equations (9) and (10), the equation (8) is satisfied by the value (7) of $K(x)$.

The value of $K(x)$ may be obtained in a more explicit form in the following manner. If we eliminate A, B, . . ., N determinantally from the equations (7) and (10), we have

$$K(x)\begin{vmatrix} \frac{1}{\alpha-p} & \frac{1}{\beta-p} & \cdots & \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & \frac{1}{\beta-q} & \cdots & \frac{1}{\nu-q} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\alpha-v} & \frac{1}{\beta-v} & \cdots & \frac{1}{\nu-v} \end{vmatrix} = -e^{\alpha x}\begin{vmatrix} 1 & \frac{1}{\beta-p} & \cdots & \frac{1}{\nu-p} \\ 1 & \frac{1}{\beta-q} & \cdots & \frac{1}{\nu-q} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{1}{\beta-v} & \cdots & \frac{1}{\nu-v} \end{vmatrix}$$

$$-e^{\beta x}\begin{vmatrix} \frac{1}{\alpha-p} & 1 & \cdots & \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & 1 & \cdots & \frac{1}{\nu-q} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\alpha-v} & 1 & \cdots & \frac{1}{\nu-v} \end{vmatrix} - \ldots$$

The determinants which occur in this equation are of the kind known as *alternants*, and may be factorised by known methods.* Performing the factorisation, we have

$$K(x) = -\frac{(\alpha-p)(\alpha-q)(\alpha-r)\ \ldots\ (\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\ \ldots\ (\alpha-\nu)}e^{\alpha x}$$
$$-\frac{(\beta-p)(\beta-q)\ldots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}\ldots-\frac{(\nu-p)(\nu-q)(\nu-r)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}.$$

Combining our results, we have the following theorem:

The solution of the integral equation

$$\phi(x) + \int_0^x \phi(s)\kappa(x-s)ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be given numerically and

---

* The evaluation of alternants of this type is due to Cauchy, *Exercices d'analyse*, **2** (1841), p. 151.

to have been expressed approximately by Prony's method in the form

$$\kappa(x) = \mathrm{P}e^{px} + \mathrm{Q}e^{qx} + \ldots + \mathrm{V}e^{vx},$$

is

$$\phi(x) = f(x) - \int_0^x \mathrm{K}(x-s)f(s)ds,$$

where

$$\mathrm{K}(x) = -\frac{(\alpha-p)(\alpha-q)\ldots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x} - \frac{(\beta-p)(\beta-q)\ldots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x} - \ldots$$
$$- \frac{(\nu-p)(\nu-q)\ \ldots\ (\nu-v)}{(\nu-\alpha)(\nu-\beta)\ \ldots\ (\nu-\mu)}e^{\nu x},$$

and where $\alpha, \beta, \gamma, \ldots, \nu$ *are the roots of the algebraic equation in* x,

$$\frac{\mathrm{P}}{x-p} + \frac{\mathrm{Q}}{x-q} + \frac{\mathrm{R}}{x-r} + \ldots + \frac{\mathrm{V}}{x-v} + 1 = 0.$$

The solution of the integral equation is thus obtained in a finite form which admits of computation.

See also Prasad, *Proc. Edin. Math. Soc.* **42** (1924) 46.

(iv.) *Integral Equations of Fredholm's and Hilbert's Type.*—The numerical solution of integral equations of the type

$$\phi(x) - \lambda\int_0^1 \kappa(x, s)\phi(s)ds = f(x),$$

where $\kappa(x, s)$ and $f(x)$ are given functions, λ is a constant, and $\phi(x)$ is the unknown function, has been discussed by Bateman, *Proc. R.S.* **100** (1921), p. 441, who gives references to the literature of the question. See also F. Tricomi, *Lincei Rend.* **33**$_1$ (1924), p. 483, **33**$_2$ (1924), p. 26, and Nyström, *Acta. Math.* **54** (1930) 185.

184. The Rayleigh-Ritz Method for Minimum Problems.—A great many problems in mathematical physics involve the determination of an unknown function to satisfy the condition that a given integral has a minimum value. Suppose for simplicity that there is only one independent variable x, and let the unknown function $y(x)$ be required to be such as to make

$$\mathrm{J} = \int_a^b f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\right)dx \tag{1}$$

a minimum, and also possibly satisfy certain further conditions.

To solve this problem, Rayleigh in 1870* and subsequent years† devised the following method, which was afterwards elaborated in a celebrated memoir by W. Ritz.‡

Let the unknown function be expanded as an infinite series of known functions with arbitrary coefficients (such as a Fourier series), say

$$y(x) = a_0\psi_0(x) + a_1\psi_1(x) + a_2\psi_2(x) + \dots \qquad (2)$$

where $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ... are functions each of which satisfies all the further conditions (e.g. boundary-conditions) which are imposed on the function $y(x)$. Then evidently a theoretical process might be imagined in which the function $y(x)$ and its derivatives are replaced in (1) by their values as furnished by (2), so that the integral J becomes a known function of the infinite set of coefficients $a_0, a_1, a_2, \dots$; and the problem would then become, to determine $a_0, a_1, a_2, \dots$ so as to make J a minimum. The validity of such a process would of course need a careful examination; but Rayleigh's proposal was to obtain an *approximate* solution by truncating the series (2) so as to retain only a finite number of coefficients $a_0, a_1, \dots, a_n$; then substituting the truncated series in (1), J is obtained as a function of $a_0, a_1, a_2, \dots, a_n$, and the values of $a_0, a_1, \dots, a_n$ can be determined from the conditions that J should be a minimum, namely

$$\frac{\partial J}{\partial a_0} = 0, \quad \frac{\partial J}{\partial a_1} = 0, \quad \dots, \quad \frac{\partial J}{\partial a_n} = 0.$$

Thus the approximate value of $y(x)$ is determined.

The same principle can be applied when J is a multiple integral taken over a domain of any number of independent variables.§

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* In finding the correction for the open end of an organ-pipe, *Phil. Trans.* 161 (1870), p. 77.

† Many examples are to be found in Rayleigh's *Theory of Sound*, e.g. §§ 88, 89, 90, 91, 182, 209, 210, 265; also *Phil. Mag.* 47 (1899), p. 556.

‡ *J. für Math.* 135 (1908), p. 1.

§ Cf. N. Kryloff, *Les méthodes de solution approchée des problèmes de la physique mathématique* (Paris, Gauthier-Villars, 1931). G. Temple and W. G. Bickley, *Rayleigh's Principle and its Applications to Engineering* (London, Oxford U. Press, 1933). H. M. James, " Some applications of the Rayleigh-Ritz method

185. Application to the Determination of Eigenvalues.—As an illustration of the Rayleigh-Ritz method, we shall solve the following problem.

The differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0 \tag{1}$$

(where λ is independent of x) admits a solution which vanishes both when $x = 1$ and when $x = -1$, only if λ has one of a certain set of special values.* These values of λ are called *proper values*, or *characteristic values*, or *autovalues*, or *eigenvalues*. We shall now show how they may be calculated numerically.

The differential equation (1) arises in the Calculus of Variations as the condition that the integral

$$J = \int_{-1}^{1} \left\{ \left(\frac{dy}{dx}\right)^2 - \lambda y^2 \right\} dx \tag{2}$$

should have a stationary value.

Let us take, as an approximation to y, the polynomial

$$y = (1 - x^2)(a_0 + a_1 x^2), \tag{3}$$

which satisfies the condition of vanishing when $x = 1$ and when $x = -1$, and which contains two undetermined coefficients a_0 and a_1. Substituting from (3) in (2), we have

$$J = \int_{-1}^{1} [\{2(a_1 - a_0)x - 4a_1 x^3\}^2 - \lambda(1 - x^2)^2(a_0 + a_1 x^2)^2] dx.$$

Performing the integration, this gives

$$J = \frac{8a_0^2}{3} + \frac{16a_0a_1}{15} + \frac{88a_1^2}{105} - \lambda\left(\frac{16a_0^2}{15} + \frac{32a_0a_1}{105} + \frac{16a_1^2}{315}\right).$$

---

to the theory of the structure of matter ", *Bull. Amer. Math. Soc.* 47 (1941), p. 869. Many other references are given by H. Bateman on pp. 91–94 of *Bulletin No. 92 of the National Research Council* (Washington, 1933).

* Actually the values of λ in question are $\frac{\pi^2}{4}, \frac{4\pi^2}{4}, \frac{9\pi^2}{4}, \ldots$, the corresponding solutions $y(x)$ being $\cos\frac{\pi x}{2}$, $\sin \pi x$, $\cos\frac{3\pi x}{2}$, . . . ; but we do not assume this knowledge in the test.

The conditions that J should have a stationary value, namely $\frac{\partial J}{\partial a_0} = 0$ and $\frac{\partial J}{\partial a_1} = 0$, are

$$\begin{cases} a_0\left(1 - \frac{2\lambda}{5}\right) + \frac{a_1}{5}\left(1 - \frac{2\lambda}{7}\right) = 0, \\ a_0\left(1 - \frac{2\lambda}{7}\right) + \frac{a_1}{7}\left(11 - \frac{2\lambda}{3}\right) = 0. \end{cases}$$

Eliminating the ratio $a_0 : a_1$, we have $\lambda^2 - 28\lambda + 63 = 0$, so

$$\begin{aligned} \lambda &= 14 \pm \sqrt{133} \\ &= 2{\cdot}467{,}437 \quad \text{or} \quad 25{\cdot}532{,}563. \end{aligned}$$

These are the approximations to the eigenvalues furnished by the Rayleigh-Ritz method when the approximation is made by a polynomial of the form (3). The correct values of the first and third * eigenvalues of the differential equation (1) are actually

$$\lambda = 2{\cdot}467{,}401 \quad \text{and} \quad \lambda = 22{\cdot}207.$$

Thus the Rayleigh-Ritz method, with the choice (3) for the unknown function, gives an excellent approximation to the lowest eigenvalue (correct to 5 significant figures) and a very rough approximation to the third eigenvalue. If we had chosen instead of (3) a polynomial of higher degree, we could, of course, have obtained better approximations to the lowest eigenvalues and also rough approximations to higher eigenvalues.

---

* The approximation (3), since it contains only even powers of x, yields only eigenvalues of odd order.

[*Copies of the Computation Sheets facing pages 270 and 278 may be obtained direct from the Publishers in quantities of not less than one dozen, at the price of 2s. 6d. per dozen sets. Special terms will be quoted for orders of one thousand and upwards.*]

ANSWERS TO EXAMPLES

CHAPTER I

Page.	Example.	
16	3.	$\frac{a}{4w}x^4 - \frac{3aw-2\beta}{6w}x^3 + \frac{aw^2-2\beta w+2\gamma}{4w}x^2 + \frac{\beta w^2-3\gamma w+6\delta}{6w}x + \text{const}$
	4.	(a) $\Delta^p\left(\frac{1}{n}\right) = (-1)^p \frac{p\,!}{n(n+1)\,.\,.\,.\,(n+p)}$. (b) $\Delta^p \cos nx = \left(2\sin\frac{nw}{2}\right)^p \cos\left(nx + \frac{npw}{2} + \frac{p\pi}{2}\right)$.
	5.	$3x(x-1)(x-2) + 10x(x-1) + 5x + 1$.
	6.	6.
	7.	$-x^3 - 3x^2 - 5x + 1$.
	8.	0·776737,4947.
	✓ 11.	0·781072,8886.
	✓ 12.	0·414213,562374.
	✓ 13.	875·311046,687.
	14.	$f(1{\cdot}25) = 0{\cdot}208459$, $f(1{\cdot}75) = 0{\cdot}227977$.
	15.	(a) 0·433178,483028,782. (b) 0·433265,874297,833.
	16.	9·648611,469336,24.
	17.	0·861232,11.

CHAPTER II

33	1.	$f(a, b) = -(a+b)/a^2b^2$: $f(a, b, c) = (ab+bc+ca)/a^2b^2c^2$; $f(a, b, c, d) = -(abc+abd+acd+bcd)/a^2b^2c^2d^2$.
	3.	$f(3) = 18$, $f(14) = 2548$.
	4.	(a) $x^3 + x$. (b) x^n.
	5.✓	681472.
	6. ✓	130326.
	7.	$f(9) = 810$: $f(6{\cdot}5) = 316{\cdot}875$.
	8.	(6) $x^4 + 5$. (7) $x^2 + x^3$.

CHAPTER III

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51	1.	0·433702,769524,48.
	2.	7·671672,8960.
	3.	0·433265,874297,832.
	4.	3·528273,8.

CHAPTER IV

68	1.	2·602060,0. 2·603144,4. 2·604226,1. 2·605305,1. 2·606381,4. 2·607455,1. 2·608526,1. 2·609594,4. 2·610660,2. 2·611723,3.
	3.	8387.
	4.	(*a*) 88294. (*b*) 84640.
	6.	$h=886$, $p=24·51$.
	7.	$p=14·23$, $\theta=100°·56$.
	8.	1·213411,6.
	9.	(*a*) 0·003333,333333. (*b*) 0·003311,258276.
	10.	−0·001978,867.
	11.	−0·001986.
	13.	5470524 : 147852 : 2664 : 24.

CHAPTER V

76	1.	(α) 0. (β) 121. (γ) 7/45.
	3.	(α) −19/720. (β) 278. (γ) $-\frac{1}{2}.7.6^5$.
	5.	(α) $x=p$, $y=3q$, $z=\frac{1}{2}(p+3q)$, $w=\frac{1}{2}(p-3q)$. (β) $x_1=-5$, $x_2=3/4$, $x_3=1$, $x_4=-2$, $x_5=6$.

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83	2.	0·67239.
	3.	0·25865.
86	2.	4·0644.
91	2.	0·67239.
98		4·861.
99	2.	0·120614,83.
106	2.	4·860806.
112	3.	−5, −3, −2.
	4.	−6, −4, −2.
115	2.	$2 \pm 1·732\sqrt{-1}$, 3·236, −1·236
118	2.	5·994.
120	2.	3·1415926.
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	2.	(α) $x = 8\cdot46$. (β) $x = 1\cdot19967864$.
	3.	4·546.
	4.	1·922884,15.
	5.	9·886.
	7.	$x = 158$.
	8.	(α) $-1\cdot95$, 1·70, 0·76. (β) $-2\cdot935$, 0·46, 1·474.
	9.	$-32\cdot74$.
	10.	1·381966, 3·618034.
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	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Ex. 1.													
(*a*)	62·92	-38·09	-15·50	-5·15	-10·46	1·54	-0·17	-1·06	2·21	2·65	4·42	-0·39	1·08
(*b*)		17·88	8·89	2·47	-4·98	-0·82	6·33	-5·90	2·06	1·30	0·94	-2·20	
Ex. 2.													
(*a*)	111·83	-50·41	-36·29	-12·18	-6·96	-0·52	-0·08	-3·35	-1·63	0·18	0·38	-0·28	0·25
(*b*)		9·28	6·61	-7·51	-8·01	-1·19	-2·08	-4·58	8·98	0·33	1·56	0·99	
Ex. 3.													
(*a*)	70·79	-58·16	18·35	0·43	-15·58	-2·16	-1·00	1·46	-1·17	4·08	2·90	-0·64	0·71
(*b*)		12·26	-8·57	-2·18	-4·91	-9·95	4·25	-6·13	-1·01	0·65	0·82	0·78	
Ex. 4.													
(*a*)	118·25	25·04	22·77	-1·37	-5·38	-0·70	-1·88	5·58	-0·63	5·70	3·57	-2·26	1·25
(*b*)		58·83	-32·79	7·16	1·08	-6·40	-0·60	-1·82	-7·58	2·50	-0·47	3·58	

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